Direct Diffeomorphic Reparameterization for Correspondence Optimization in Statistical Shape Modeling

—— Ph.D. Thesis Defense

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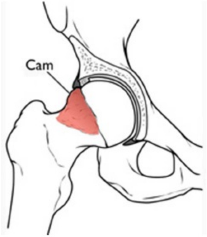
MMAE
ILLINOIS INSTITUTE OF TECHNOLOGY

Outline

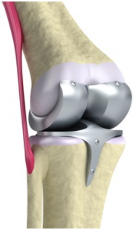
Why Statistical Shape Model (SSM)?

SSM of a shape population

1. Better understanding
   mean & typical range of shape variations
2. Compact representation
   mean + linear combination of variations
3. Classification
   Medical diagnosis etc.
4. Shape construction
   Patient-specific modeling etc.



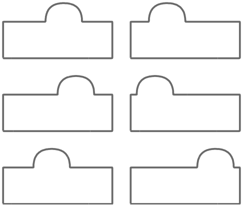
Cam impingement



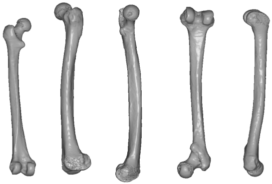
Knee implant

Figures courtesy of American Academy of Orthopaedic Surgeons (AAOS)

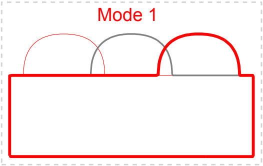
What is Statistical Shape Model?



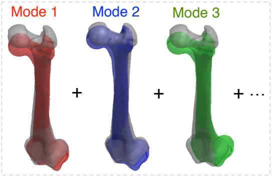
Box-bump



Femur



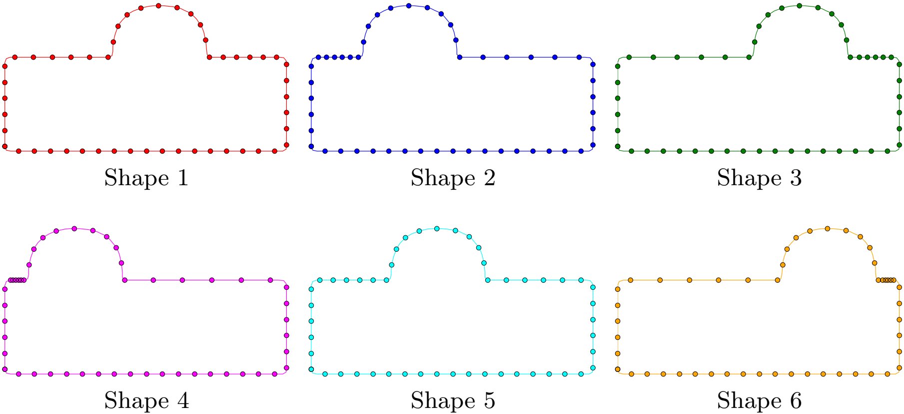
SSM



SSM

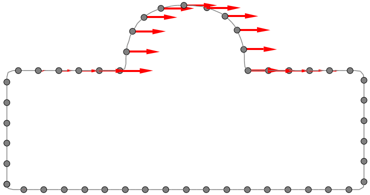
- SSM = mean + modes
- Mode: shape variation pattern

How to get SSM?

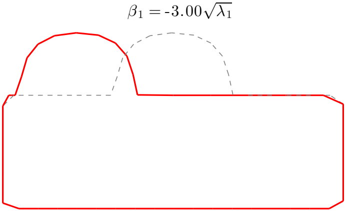


- 6 shape, each represented by 50 points (landmarks)
- The i-th shape: $\underset{100\times 1}{\mathbf{X}_i} = [x_i^{(1)}, y_i^{(1)}, x_i^{(2)}, y_i^{(2)}, ..., x_i^{(50)}, y_i^{(50)}]^T$

Statistical analysis steps



Mean $\bar{\mathbf{X}}$ and mode vector $\tilde{\mathbf{v}}_1$

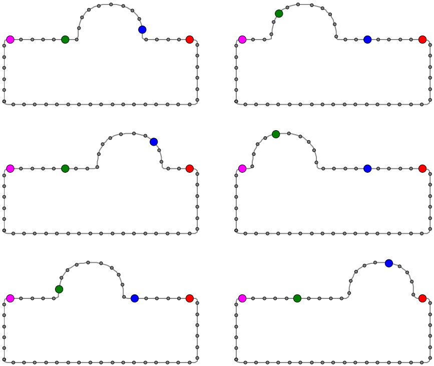


Mode 1: $\bar{\mathbf{X}} + \beta_1 \tilde{\mathbf{v}}_1$

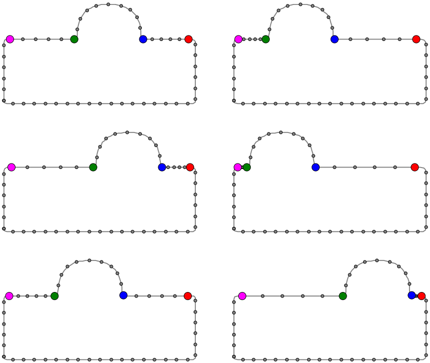
- Mean shape: $\underset{100\times 1}{\bar{\mathbf{X}}} = (\mathbf{X}_1 + \mathbf{X}_2 + ... + \mathbf{X}_6)/6$
- Shape data matrix: $\underset{100\times 6}{\mathbf{X}_c} = \left[\mathbf{X}_1 - \bar{\mathbf{X}}, \mathbf{X}_2 - \bar{\mathbf{X}}, ..., \mathbf{X}_6 - \bar{\mathbf{X}}\right]$
- Principal Component Analysis (PCA) by eigenvalue decomposition

  $\underset{6\times 6}{\mathbf{X}_c^T \mathbf{X}_c} \underset{6\times 1}{\mathbf{v}} = \lambda\, \mathbf{v} \implies$ Variations $\{\lambda_m\}$, Eigenvectors $\{\mathbf{v}_m\}(m = 1, 2, ..., 6)$
- Mode vector from eigenvector: $\underset{100\times 1}{\tilde{\mathbf{v}}_m} = \underset{100\times 6}{\mathbf{X}_c} \underset{6\times 1}{\mathbf{v}_m}$
- Mode range parameter $\beta_m \in \left[-3\sqrt{\lambda_m}, +3\sqrt{\lambda_m}\right]$

Shape variation pattern $\sim$ shape correspondence

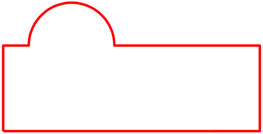


Poor correspondence



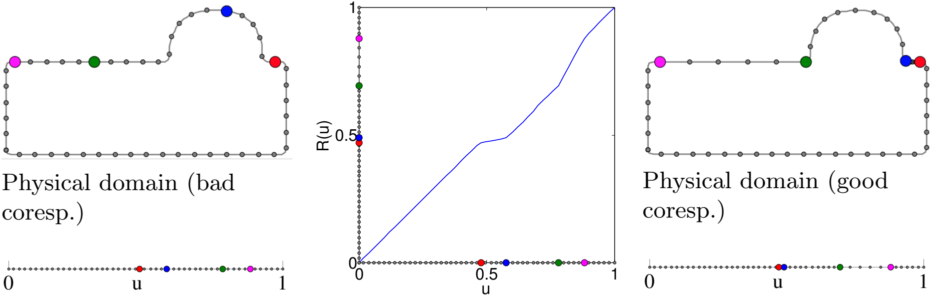
Good correspondence

Unreasonable variation



Reasonable variation

Manipulate correspondence by reparametrization



Physical domain (bad coresp.)

Parameter domain

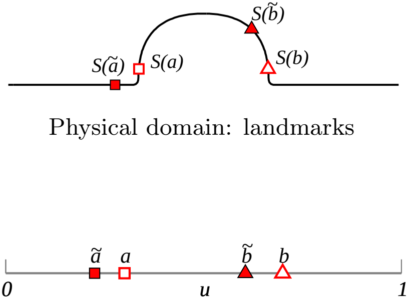
Reparametrization function

Physical domain (good coresp.)

Parameter domain

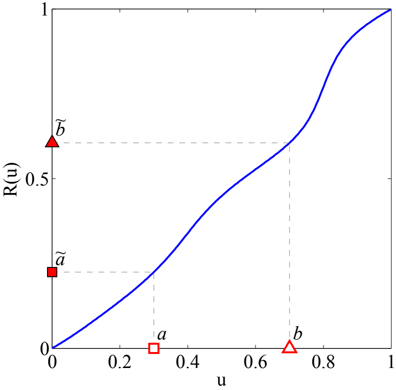
- Shape 6 as an example
- Shape parametrization (physical $\overset{\text{mapping}}{\Longleftrightarrow}$ parameter) required
- Correspondence manipulation $\sim$ landmarks redistribution
- Landmarks redistribution achieved by reparametrization

Correspondence manipulation on curve by $R(u)$



Physical domain: landmarks

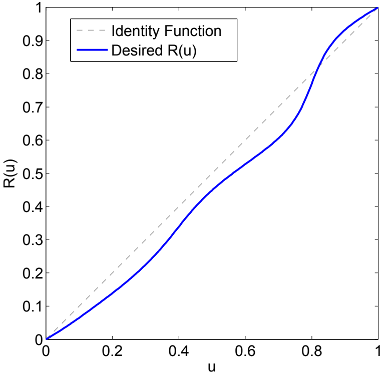
Parameter domain: landmark points



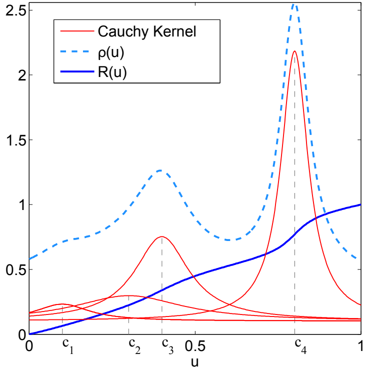
Raparameterization function $R(u)$

- Correspondence manipulation $\sim$ Landmark redistribution
- Shape parametrized by $S(u)$, reparametrized by $R(u)$
- Landmark $S(u)$ moves to $S[R(u)]$

Reparametrization by concatenation: curve case



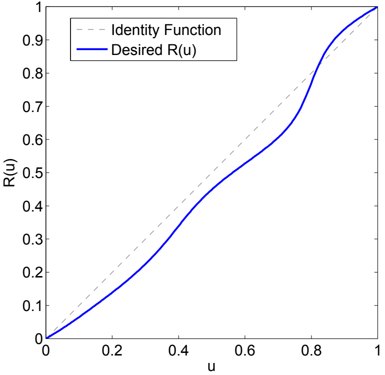
$R(u)$ to model



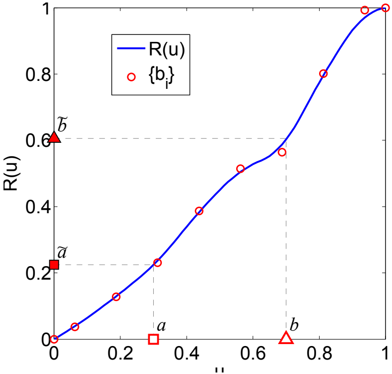
Composition of 4 Cauchy kernels

- Conventional reparametrization: sequential concatenations
- Local reparametrization: Cauchy kernel
- Cumbersome and inefficient

Proposed direct reparametrization: curve case



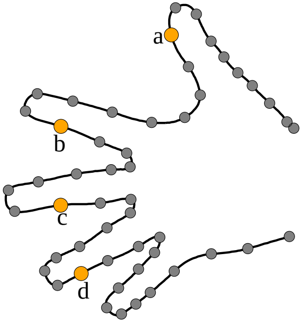
$R(u)$ to model



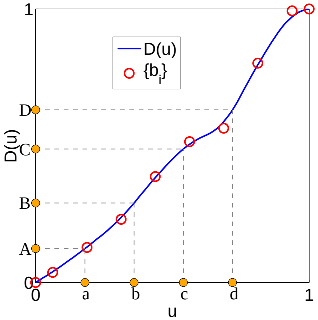
Single B-spline function

- $R(u) = \sum_{i=0}^{n} B_{i,p}(u) b_i, \qquad 0 \leq u \leq 1;\ i = 0, ..., n$
- Diffeomorphic constraint: $b_i - b_{i+1} \leq 0, \qquad i = 0, 1, ..., n-1.$

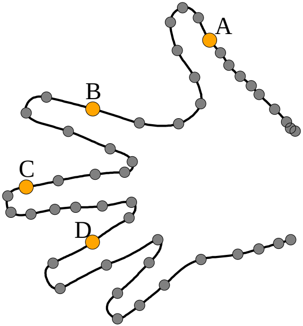
A reparametrization example



Landmarks before reparam.



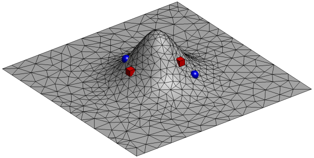
Reparmeterization function



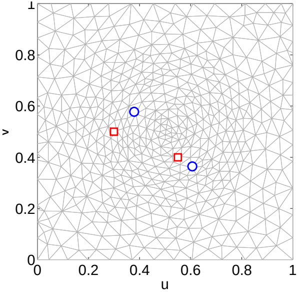
Landmarks after reparam.

- More complex geometry: hand contour
- Arbitrary landmarks redistribution achievable

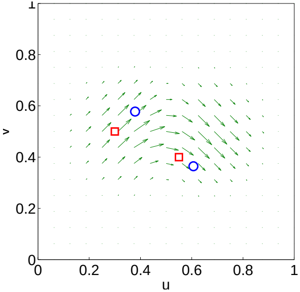
Correspondence manipulation on surface by $\mathbf{R}(u, v)$



Physical domain: landmarks



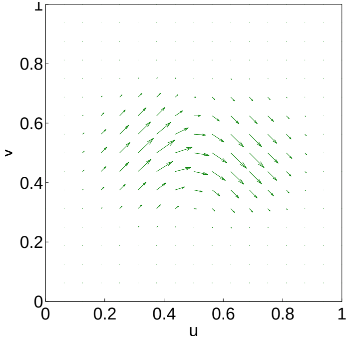
Parameter domain: landmark points



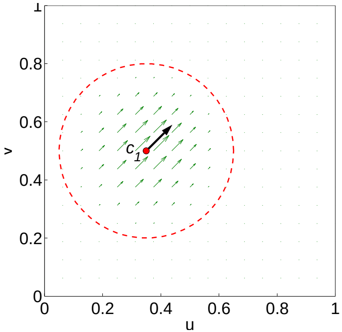
Raparameterization function $\mathbf{R}(u, v)$

- Correspondence manipulation $\sim$ Landmark redistribution
- Shape parametrized by $\mathbf{S}(u, v)$
- Precisely controlled by reparametrization function $\mathbf{R}(u, v)$

Reparametrization by concatenation: surface case

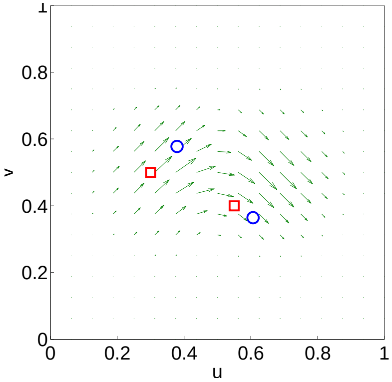


$\mathbf{R}(u, v)$ desired

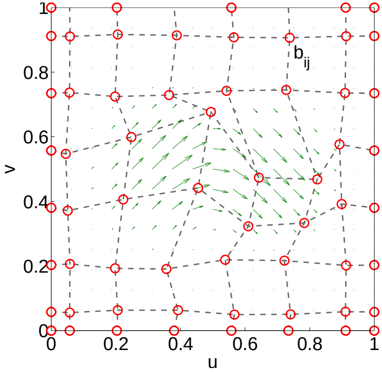


- Conventional reparametrization: sequential concatenations
- Local reparametrization: Clamped Plate Spline warp
- Cumbersome and inefficient

Proposed direct reparametrization: surface case



$\mathbf{R}(u,v)$ to model

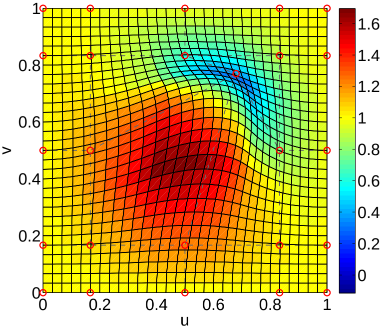


By a single 8×8 B-spline

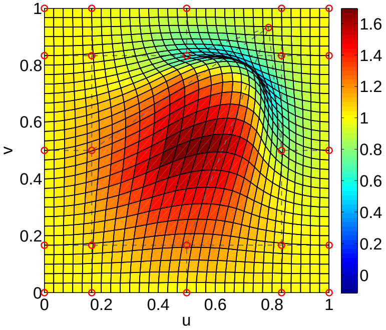
$$R^u(u,v) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} B_{i,p}(u) B_{j,q}(v) b^u_{i,j};\ R^v(u,v) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} B_{i,p}(u) B_{j,q}(v) b^v_{i,j}$$

$0 \le u, v \le 1;\ i = 0, ..., n_1;\ j = 0, ..., n_2$

Diffeomorphic constraint: Jacobian positivity



Valid deformation



Invalid deformation

$$J(\mathbf{u}) = \begin{vmatrix} \frac{\partial R^u(u,v)}{\partial u} & \frac{\partial R^u(u,v)}{\partial v} \\ \frac{\partial R^v(u,v)}{\partial u} & \frac{\partial R^v(u,v)}{\partial v} \end{vmatrix} > 0, \quad \forall (u,v) \in [0,1].$$

Correspondence optimization formulation

- Description Length (DL): the message length for transmitting statistical model: $\bar{\mathbf{x}}$, $\{\mathbf{v}_m\}$ and $\{\lambda_m\}$
- Simplified version as objective function

$$f \doteq DL = \sum_{m=1}^{n_S-1} L_m$$

$$\text{where} \quad L_m = \begin{cases} 1 + \log(\lambda_m/\lambda_{\text{cut}}) & \lambda_m \geq \lambda_{\text{cut}} \\ \lambda_m/\lambda_{\text{cut}} & \text{otherwise} \end{cases}$$

- Optimization formulation

$$\min_{\mathbf{b}} \quad f(\mathbf{b}) = \sum_{\lambda_i \geq \lambda_{\text{cut}}} \left[1 + \log \frac{\lambda_k(\mathbf{b})}{\lambda_{\text{cut}}}\right] + \sum_{\lambda_k < \lambda_{\text{cut}}} \frac{\lambda_k(\mathbf{b})}{\lambda_{\text{cut}}} \tag{1a}$$

$$\text{s.t.} \quad \left[\mathbf{C}^T(\mathbf{b})\mathbf{C}(\mathbf{b})\right] \mathbf{v}_k(\mathbf{b}) = \lambda_i(\mathbf{b})\mathbf{v}_k(\mathbf{b}) \tag{1b}$$

$$\mathbf{v}_k^T(\mathbf{b})\mathbf{v}_k(\mathbf{b}) = 1, \quad k = 1, \ldots, n_S \tag{1c}$$

$$g(\mathbf{b}) < 0 \tag{1d}$$

Direct method

- Eigen decomposition form:
  $\mathbf{C}^T\mathbf{C}\,\mathbf{v}_m = \lambda_m\,\mathbf{v}_m \quad (1); \quad \mathbf{v}_m^T\mathbf{v}_m = 1 \quad (2)$
- Consider sensitivity for $f(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \mathbf{V})$

$$\frac{\mathrm{d}\, f(\boldsymbol{\alpha})}{\mathrm{d}\, b_r} = \frac{\partial f}{\partial b_r} + \sum_{m=1}^{n_S} \frac{\partial f}{\partial \lambda_m}\frac{\partial \lambda_m}{\partial b_r} + \sum_{m=1}^{n_S} \left(\frac{\partial f}{\partial \mathbf{v}_m}\right)^T \frac{\partial \mathbf{v}_m}{\partial b_r} = \sum_{m=1}^{n_S} \frac{\partial f(\boldsymbol{\lambda})}{\partial \lambda_m}\frac{\partial \lambda_m}{\partial b_r}$$

- Differentiating (1) and (2) w.r.t b_r for each q gives

$$\left(\lambda_m \mathbf{I}_{n_S} - \mathbf{C}^T\mathbf{C}\right)\frac{\partial \mathbf{v}_m}{\partial b_r} + \mathbf{v}_m \frac{\partial \lambda_m}{\partial b_r} = \frac{\partial(\mathbf{C}^T\mathbf{C})}{\partial b_r}\mathbf{v}_m \qquad \mathbf{v}_m^T\frac{\partial \mathbf{v}_m}{\partial b_r} = 0$$

- Direct sensitivity: solve for $\frac{\partial \lambda_m}{\partial b_r}$, $\frac{\partial \mathbf{v}_m}{\partial b_r}$ directly and plug into (3)

$$\begin{bmatrix} 0 & \mathbf{v}_m^T \\ \mathbf{v}_m & \lambda_m \mathbf{I}_{n_S} - \mathbf{C}^T\mathbf{C} \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda_m}{\partial b_r} \\ \frac{\partial \mathbf{v}_m}{\partial b_r} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\partial(\mathbf{C}^T\mathbf{C})}{\partial b_r}\mathbf{v}_m \end{bmatrix}$$

Adjoint method

- Introduce Lagrangian with (1) and (2) as constraints

$$\mathcal{L} = f(\boldsymbol{\lambda}) + \sum_{m=1}^{n_S} \boldsymbol{\mu}_m^T \left(\mathbf{C}^T\mathbf{C}\,\mathbf{v}_m - \lambda_m \mathbf{v}_m\right) + \sum_{m=1}^{n_S} \eta_m (\mathbf{v}_m^T \mathbf{v}_m - 1) \tag{4}$$

  Adjoint variables $\boldsymbol{\mu}_m$ and η_m are just Lagrange multipliers

- Differentiation of (4) w.r.t b_r gives

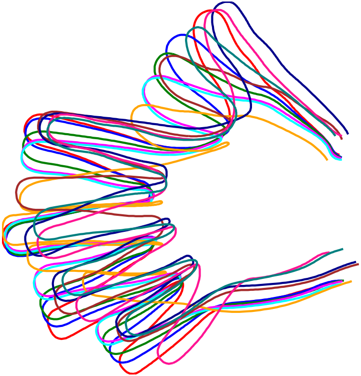
$$\begin{aligned}\frac{\mathrm{d}\, f}{\mathrm{d}\, b_r} &= \sum_{m=1}^{n_S} \boldsymbol{\mu}_m^T \frac{\partial(\mathbf{C}^T\mathbf{C})}{\partial b_r}\mathbf{v}_m + \sum_{m=1}^{n_S} \left(\frac{\partial f}{\partial \lambda_m} - \boldsymbol{\mu}_m^T \mathbf{v}_m\right)\frac{\partial \lambda_m}{\partial b_r} \\ &+ \sum_{m=1}^{n_S} \left[\boldsymbol{\mu}_m^T(\mathbf{C}^T\mathbf{C} - \lambda_m \mathbf{I}_{n_S}) + 2\eta_m \mathbf{v}_m^T\right]\frac{\partial \mathbf{v}_m}{\partial b_r}\end{aligned} \tag{5}$$

- Adjoint sensitivity: solve (6) for $\boldsymbol{\mu}_m$ and η_m and plug into (5)

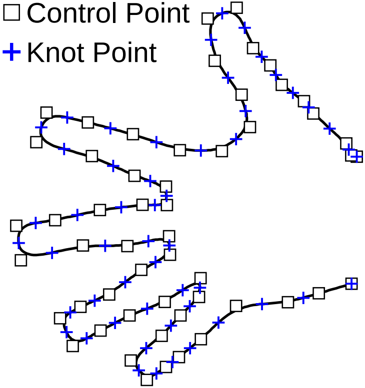
$$\begin{bmatrix} \mathbf{v}_m^T & 0 \\ \mathbf{C}^T\mathbf{C} - \lambda_m \mathbf{I}_{n_S} & 2\mathbf{v}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_m \\ \eta_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \lambda_m} \\ \mathbf{0} \end{bmatrix} \tag{6}$$

  Adjoint equation* (6) avoids the need to compute $\frac{\partial \lambda_m}{\partial b_r}$, $\frac{\partial \mathbf{v}_m}{\partial b_r}$

Experiment 1 (curve): hand



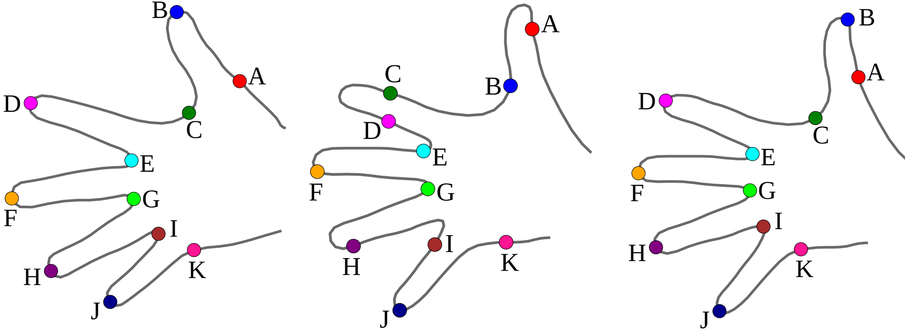
40 hand contours



B-spline representation

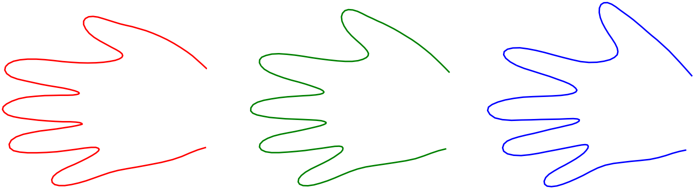
- Data courtesy of Technical University of Denmark

Correspondence before vs. after optimziation

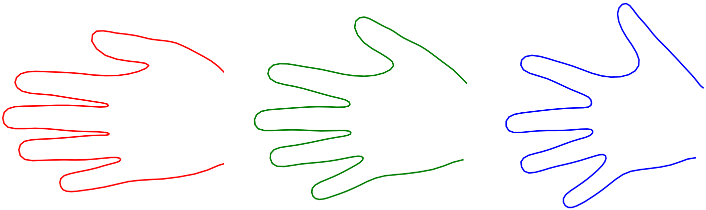


Shape 1 (fixed) Initial: $\mathbf{S}_{21}(u)$ Optimized: $\mathbf{S}_{21}(R(u))$

Statistical mode before vs. after optimization

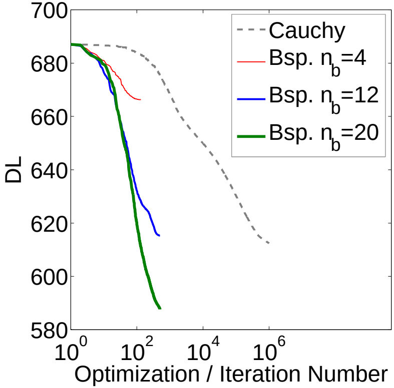


$\beta_1 = -3\sqrt{\lambda_1}$ $\beta_1 = 0$ (mean) $\beta_1 = +3\sqrt{\lambda_1}$

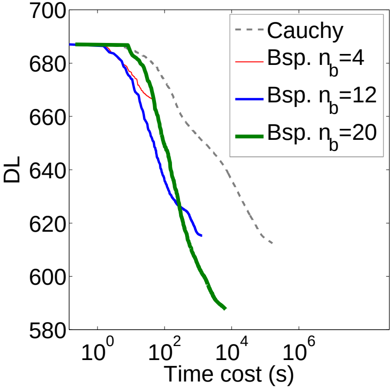


$\beta_1 = -3\sqrt{\lambda_1}$ $\beta_1 = 0$ (mean) $\beta_1 = +3\sqrt{\lambda_1}$

Time cost comparison: concatenation vs. direct reparam.

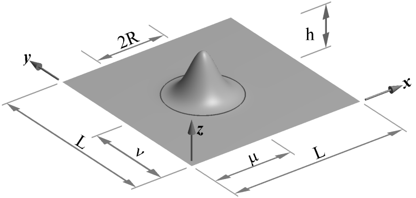


DL w.r.t iterations

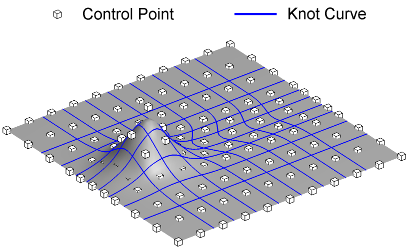


DL w.r.t time cost

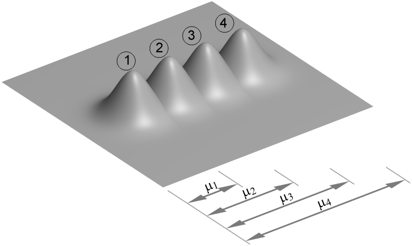
Experiment 2 (surface): plane-bump



Plane-bump geometry

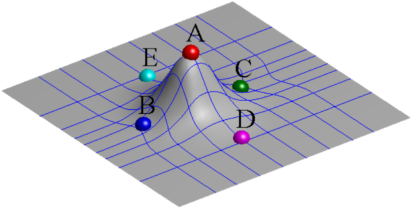


B-spline represented plane-bump

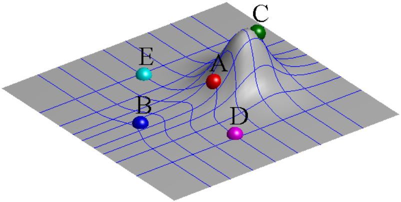


4 instances superimposed

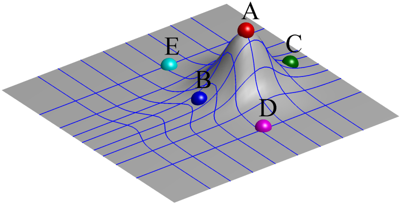
Correspondence before vs. after optimziation



5 feature points fixed on Shape 2

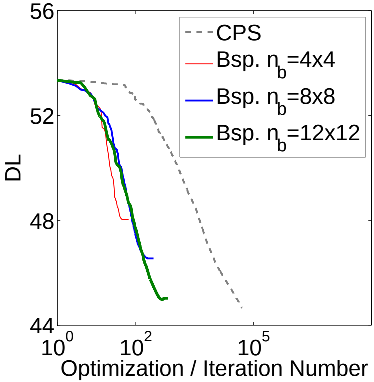


Corresponding points on Shape 4 before optimization

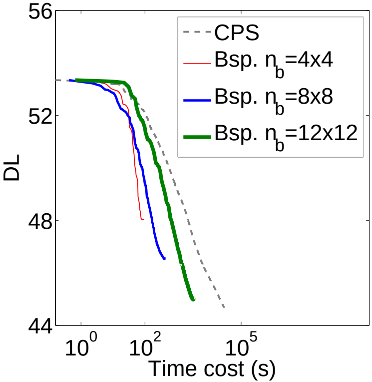


Corresponding points on Shape 4 after optimization

Time cost comparison: concatenation vs. direct reparam.

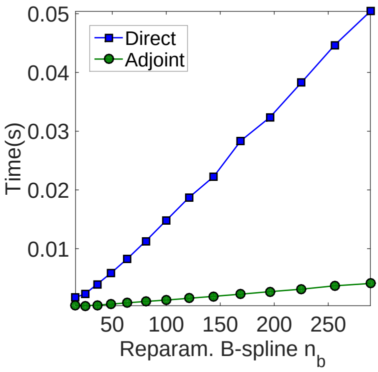


DL w.r.t iterations

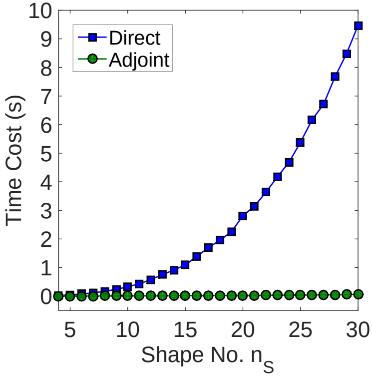


DL w.r.t time cost

Time complexity of analytical gradient



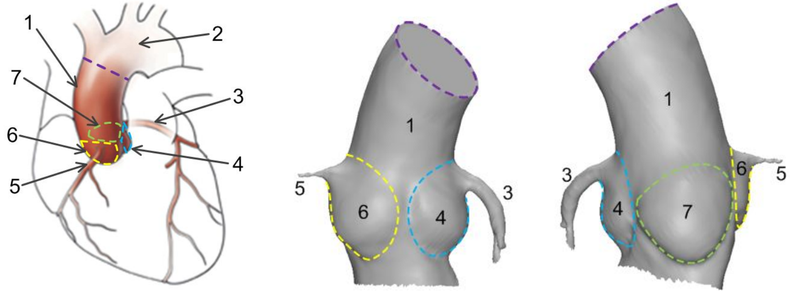
Vary $n_b = n_{b1} n_{b2}$



Vary n_S

- Adjoint sensitivity scales better w.r.t n_b and n_S

Aorta anatomy

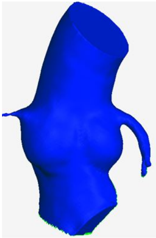


Ascending aorta (anterior view) | Front view | Back view

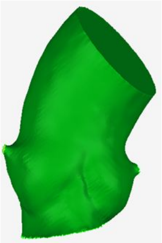
- 1) Ascending aorta; 2) Aortic arch;
- 3) Left coronary artery; 5) Right coronary artery;
- 4) Left coronary sinus; 6) Right coronary sinus; 7) Non-coronary sinus

Input raw triangle meshes

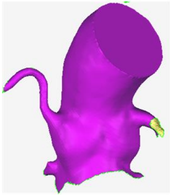
Shape 1



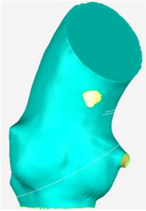
Shape 2



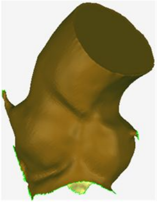
Shape 3



Shape 4



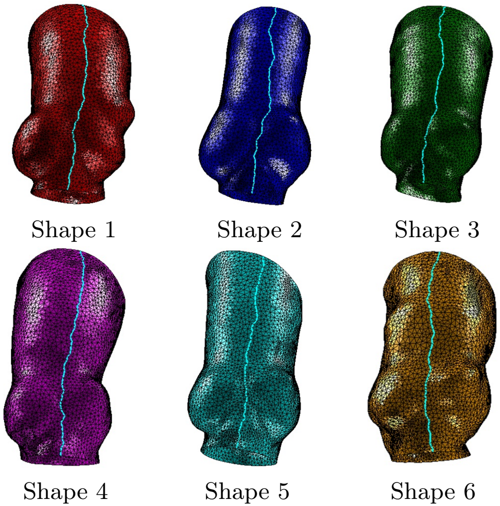
Shape 5



Shape 6

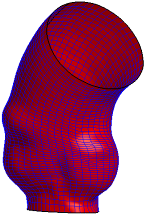
- From CT images
- 6 patients
- Shape 3 with severe aneurysm
- Courtesy of Georgia Tech

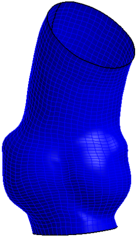
Generatrix construction

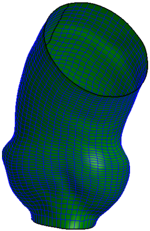


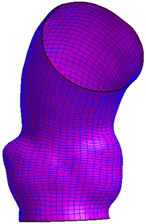
- Consistent cylindrical topology

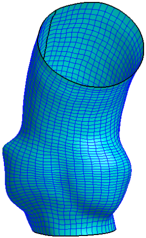
Training set B-splines fitting

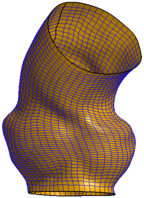

Shape 1


Shape 2

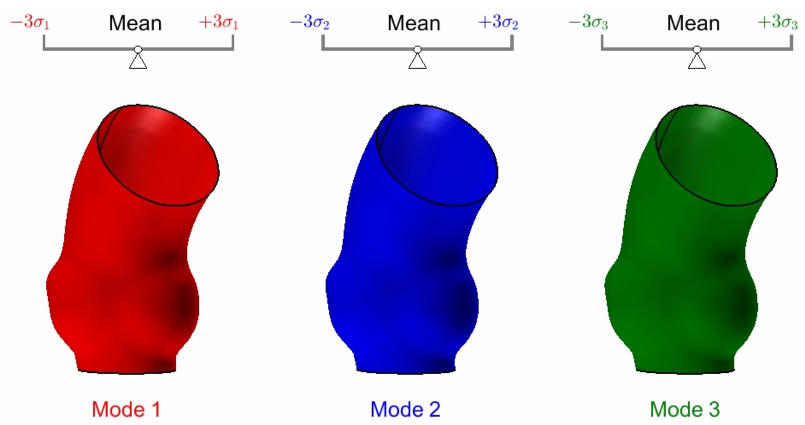

Shape 3


Shape 4


Shape 5

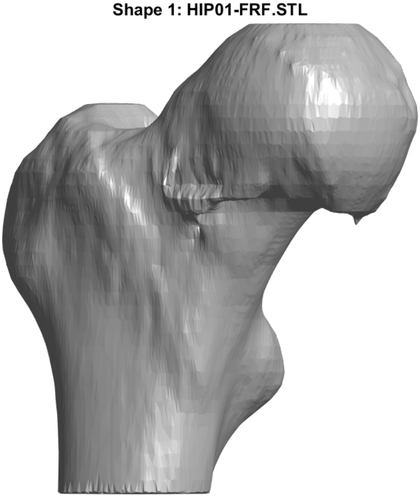

Shape 6

Statistical modes analysis



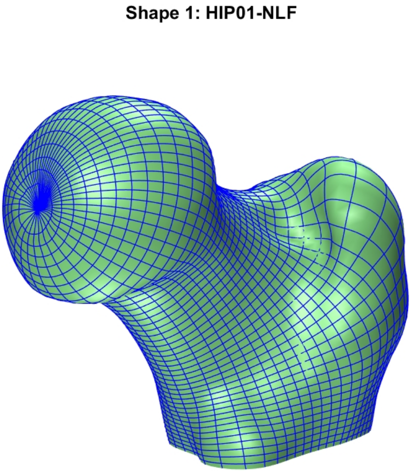
- Mode 1: Ascending aorta dilation, $\lambda_1 = 2.06$ (64.5%)
- Mode 2: Coronary sinus dilation, $\lambda_2 = 0.44$ (13.8%)
- Mode 3: Annulus dilation, $\lambda_3 = 0.38$ (11.8%)

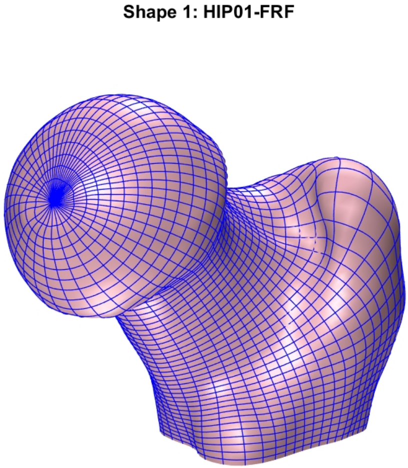
Raw proximal femur data



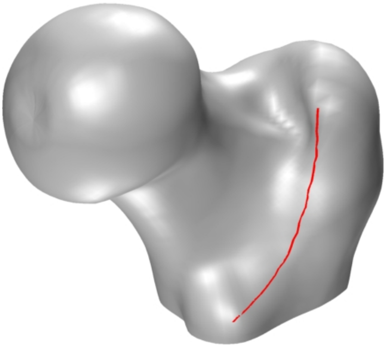
- 29 proximal femur meshes
- 17 patients
- 13 healthy, 16 unhealthy
- Courtesy of RUSH hospital

Two training sets: healthy and unhealthy

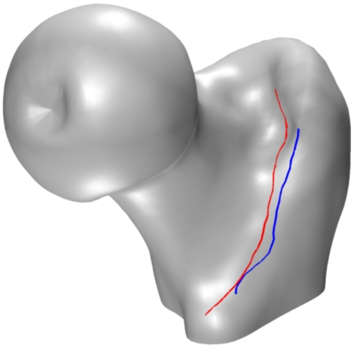




Correspondence improvement



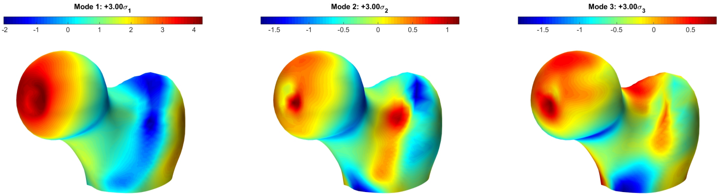
Reference shape feature line 1



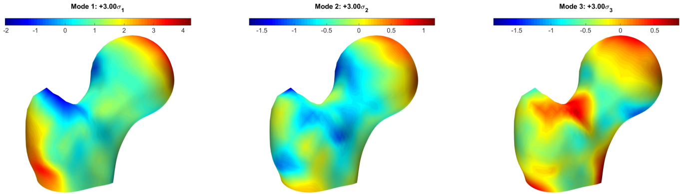
A non-ref shape corresponding line 1 before (red) and after (blue) optimization

- Feature line: intertrochanteric line
- Improvement noticeable at the greater trochanter eminence
- Similar improvement observed in unhealthy group

Statistical modes analysis: healthy group

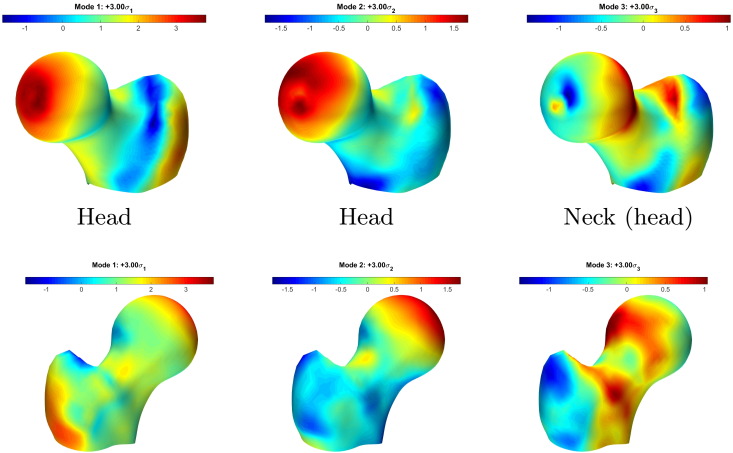


Head Neck valley Neck (trochanter)



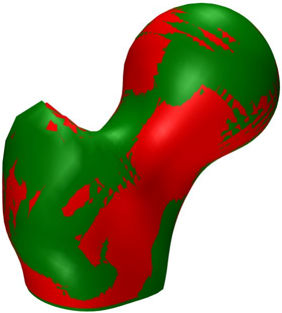
- Mode 1: femur head ball enlarging and the trochanter shrinking (4.3mm)
- Mode 2: additional thickening pattern of intertrochanteric valley (1.2 mm)
- Mode 3: bone material depositing at femur neck near trochanter (0.7mm)

Statistical modes analysis: unhealthy group

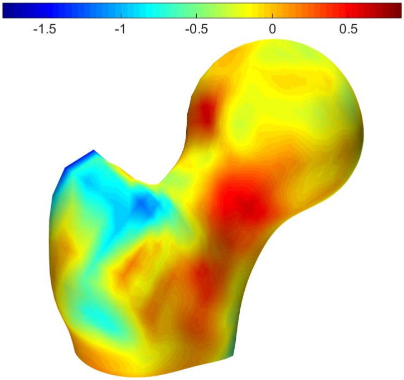


- Mode 1: femur head ball enlarging and the trochanter shrinking (3.9mm)
- Mode 2: femur head ball enlarging (1.7 mm)
- Mode 3: bone material depositing at femur neck near head (1mm)

Statistical modes analysis: unhealthy group



Superimposed



Shell distance field

- Bone material (0.7mm) of unhealthy mean on unhealthy mean
- Probable cause of cam impingement among unhealthy group

Discrete covariance matrix

- Suppose training set of n_S shapes $\{\mathcal{S}_i\}$ $(i = 1, 2, ..., n_S)$
  - Landmark: $\mathbf{x}_i^{(j)} = \left[x_i^{(j)}, y_i^{(j)}, z_i^{(j)}\right] \in \mathcal{S}_i \quad (j \in 1, 2, ..., n_P)$
  - Shape vector: $\mathbf{X}_i \doteq \left[\mathbf{x}_i^{(1)}, \mathbf{x}_i^{(2)}, ..., \mathbf{x}_i^{(n_P)}\right]^T$
- Mean shape: $\bar{\mathbf{X}} \doteq \frac{1}{n_S} \sum_{i=1}^{n_S} \mathbf{X}_i$.
- Shape data matrix: $\mathcal{X} = [\mathbf{X}_1 - \bar{\mathbf{X}},\ \mathbf{X}_2 - \bar{\mathbf{X}},\ ...,\ \mathbf{X}_{n_S} - \bar{\mathbf{X}}]$
- **Discrete covariance matrix**

$$\widetilde{\mathbf{D}} \doteq \frac{1}{n_S - 1} \sum_{i=1}^{n_S} \left(\mathbf{X}_i - \bar{\mathbf{X}}\right)^T \left(\mathbf{X}_i - \bar{\mathbf{X}}\right) = \frac{1}{n_S - 1} \mathcal{X}^T \mathcal{X}$$

Or entry-wise definition: $\widetilde{D}_{i_1 i_2} \doteq \frac{1}{n_S - 1} \left(\mathbf{X}_{i_1} - \bar{\mathbf{X}}\right)^T \left(\mathbf{X}_{i_2} - \bar{\mathbf{X}}\right)$

- PCA for SSM variation $\{\lambda_m\}$ and modes $\{\mathbf{v}_m\}$

$$\widetilde{\mathbf{D}}\, \mathbf{v}_m = \lambda_m \mathbf{v}_m$$

Continuous covariance matrix

- Mean shape: $\bar{\mathbf{S}}(\mathbf{u}) \doteq \frac{1}{n_S} \sum_{i=1}^{n_S} \mathbf{S}_i(\mathbf{u})$.
- **Continuous formulation *I***

$$C^I_{i_1 i_2} \doteq \frac{1}{n_S - 1} \int_{\mathcal{U}} \left[\mathbf{S}_{i_1}(\mathbf{u}) - \bar{\mathbf{S}}(\mathbf{u})\right]^T \left[\mathbf{S}_{i_2}(\mathbf{u}) - \bar{\mathbf{S}}(\mathbf{u})\right] \,\mathrm{d}\mathbf{u}$$

- **Continuous formulation *II***

$$C^{II}_{i_1 i_2} \doteq \frac{\int_{\mathcal{U}} \left[\mathbf{S}_{i_1}(\mathbf{u}) - \bar{\mathbf{S}}(\mathbf{u})\right]^T \left[\mathbf{S}_{i_2}(\mathbf{u}) - \bar{\mathbf{S}}(\mathbf{u})\right] |\mathbf{J}(\mathbf{u})| \,\mathrm{d}\mathbf{u}}{(n_S - 1) \int_{\mathcal{U}} |\mathbf{J}(\mathbf{u})| \,\mathrm{d}\mathbf{u}},$$

  - Curve: $|\mathbf{J}(u)| = \left|\frac{\mathrm{d}\,\bar{\mathbf{S}}(\mathbf{u})}{\mathrm{d}\,u}\right|$
  - Surface: $|\mathbf{J}(u,v)| = \left|\frac{\partial\,\bar{\mathbf{S}}(u,v)}{\partial u} \times \frac{\partial\,\bar{\mathbf{S}}(u,v)}{\partial v}\right|$

Continuous formulation I with analytical integral

- Bézier curves: $\mathbf{S}_i(u) = \sum_{j=0}^{p} B_j^p(u)\mathbf{P}_j^{(i)},\ u \in [0,1]$

$$C_{i_1 i_2}^I = \frac{1}{n_S - 1}\int_0^1 \sum_{j=0}^{2p} B_j^{2p}(u) Q_j^{(i_1,i_2)}\, \mathrm{d}u$$

$$= \sum_{j=0}^{2p} \sum_{l=\max(0,j-p)}^{\min(j,p)} \frac{\binom{p}{l}\binom{p}{j-l}\widehat{\mathbf{P}}_l^{(i_1)T}\widehat{\mathbf{P}}_{j-l}^{(i_2)}}{\binom{2p}{j}(n_S-1)(2p+1)}.$$

- Bézier surfaces: $\mathbf{S}_i(\mathbf{u}) = \sum_{j=0}^{p}\sum_{k=0}^{q} B_j^p(u)B_k^q(v)\mathbf{P}_{j,k}^{(i)},\ u \times v \in [0,1]^2$

$$C_{i_1 i_2}^I = \sum_{j=0}^{2p}\sum_{k=0}^{2q}\sum_{l=\max(0,j-p)}^{\min(j,p)}\sum_{m=\max(0,k-q)}^{\min(k,q)} \frac{\binom{p}{l}\binom{p}{j-l}\binom{q}{m}\binom{q}{k-m}\widehat{\mathbf{P}}_{l,m}^{(i_1)T}\widehat{\mathbf{P}}_{j-l,k-m}^{(i_2)}}{\binom{2p}{j}\binom{2q}{k}(n_S-1)(2p+1)(2q+1)}$$

- B-spline curves/surfaces: summation over knot spans

Continuous formulation II with analytical integrand

- Bézier curves

$$C^{II}_{i_1 i_2} \frac{\int_0^1 \sum_{j=0}^{2p} \sum_{k=0}^{p-1} B_j^{2p}(u) B_k^{p-1}(u) Q_j^{(i_1,i_2)} \widetilde{P}_k \, \mathrm{d}u}{(n_S - 1) \int_0^1 \sum_{k=0}^{p-1} B_k^{p-1}(u) \widetilde{P}_k \, \mathrm{d}u}$$

where $\widetilde{P}_k = p(\bar{P}_{k+1} - \bar{P}_k)$. and

$$Q_j^{(i_1,i_2)} = \sum_{l=\max(0,j-p)}^{\min(j,p)} \frac{\binom{p}{l}\binom{p}{j-l}}{\binom{2p}{j}} \widehat{\mathbf{P}}_l^{(i_1)T} \widehat{\mathbf{P}}_{j-l}^{(i_2)}$$

- Similar for Bézier surface and B-spline curves/surfaces
- Formulation I and II can be approximated by quadrature

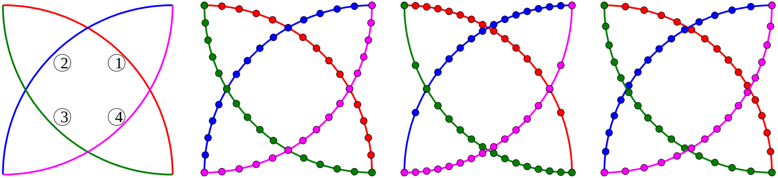
Incorporation of reparametrization

- $C^{I}_{i_1 i_2} \doteq \frac{1}{n_S - 1} \int_{\mathcal{U}} \widehat{\mathbf{S}}_{i_1}[\mathbf{R}_{i_1}(\mathbf{u})]^T \widehat{\mathbf{S}}_{i_2}[\mathbf{R}_{i_2}(\mathbf{u})] \, \mathrm{d}\mathbf{u}$

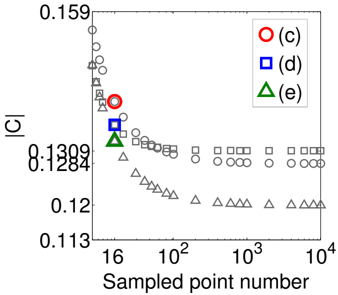
Shapes $\{\mathbf{S}_i(u)\}$		Reparam. $\{R_i(u)\}$	Gauss Pt. No.
Type	Degree	Degree	n_G^*
Bézier curve	p	—	$p+1$
Bézier surface	$p \times q$	—	$(p+1)(q+1)$
B-spline curve	p	—	$p+1$
B-spline surface	$p \times q$	—	$(p+1)(q+1)$
Bézier curve	p	d	$pd+1$
Bézier surface	$p \times q$	$d \times e$	$(pd+1)(qe+1)$
B-spline curve	p	d	$pd+1$
B-spline surface	$p \times q$	$d \times e$	—

- $C^{II}_{i_1 i_2} \doteq \dfrac{\int_{\mathcal{U}} \widehat{\mathbf{S}}_{i_1}[\mathbf{R}_{i_1}(\mathbf{u})]^T \widehat{\mathbf{S}}_{i_2}[\mathbf{R}_{i_2}(\mathbf{u})] \, |\mathbf{J}(\mathbf{R}(\mathbf{u}))| \, \mathrm{d}\mathbf{u}}{(n_S - 1) \int_{\mathcal{U}} |\mathbf{J}(\mathbf{R}(\mathbf{u}))| \, \mathrm{d}\mathbf{u}}$

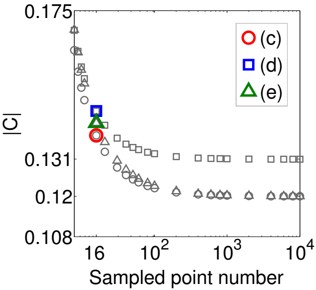
Discrete covariance matrix of qurater circles



4 quarter circles | Angle span | X-coordinate | Chordal dist.

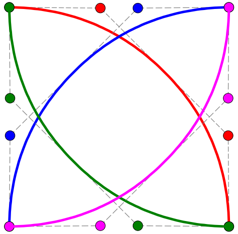


Discrete points

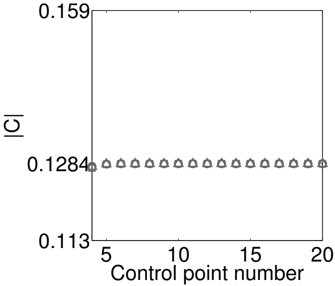


Discrete points weighted

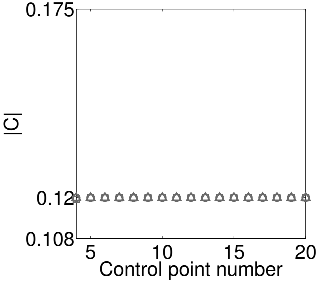
Continuous covariance matrix of qurater circles



Fitted B-splines



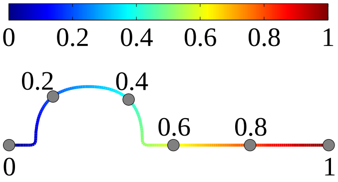
Formulation I



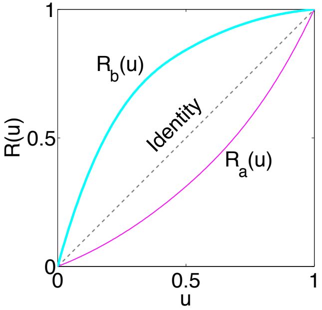
Formulation II

- Discrete formulation subject to sampling density/pattern
- Continuous formulations independent of sampling density/pattern

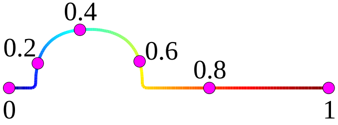
Line-bump shapes and two reparametrizations



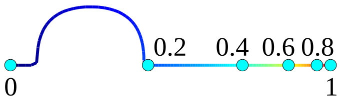
Original parameterization



Different reparameterizations

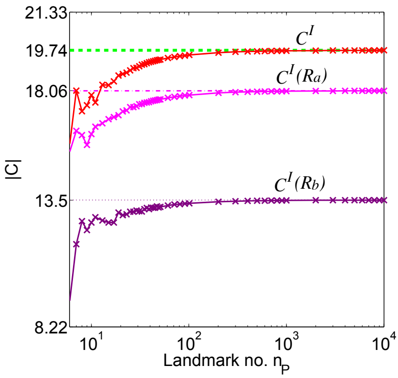


Reparameterization $R_a(u)$

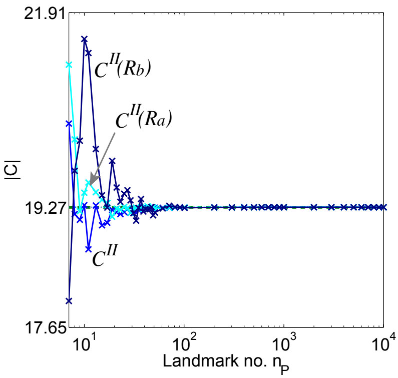


Reparameterization $R_b(u)$

Covariance matrix convergence under reparametriation



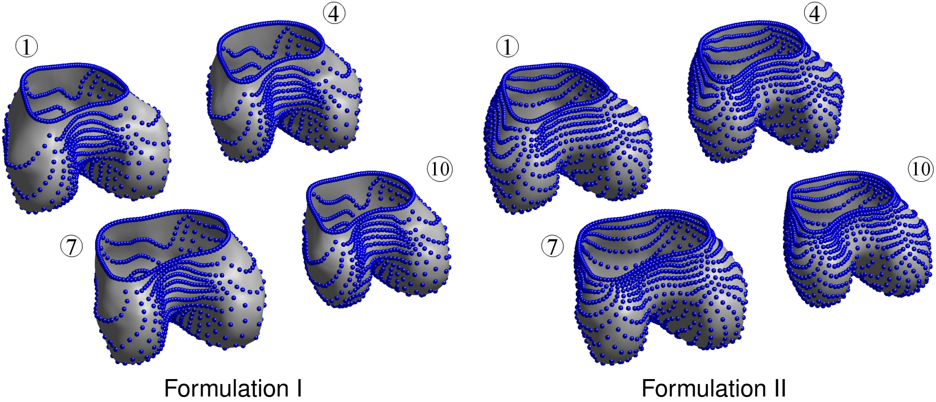
Formulation I



Formulation II

- Formulation II is independent of parametrization

Influence on correspondence optimization: C^I vs. C^{II}



- Compare optimized landmarks by using C^I and C^{II}
- Formulation II yields more reasonable landmark distribution

Summary and outlook

Thesis contributions

1. Direct diffeomorphic reparametrization achieved through B-splines
2. Full differentiability of objective function w.r.t optimization variables
3. Adjoint method for analytical gradient computation
4. Analytical form and efficient numerical quadrature for covariance matrix
5. Viable application to real medical data with valuable clinical insights

Future work

- Extension from single patch to complex topologies
- Experiments with larger data pool