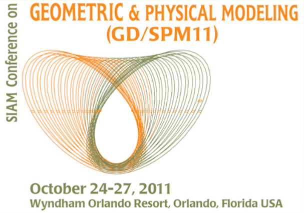
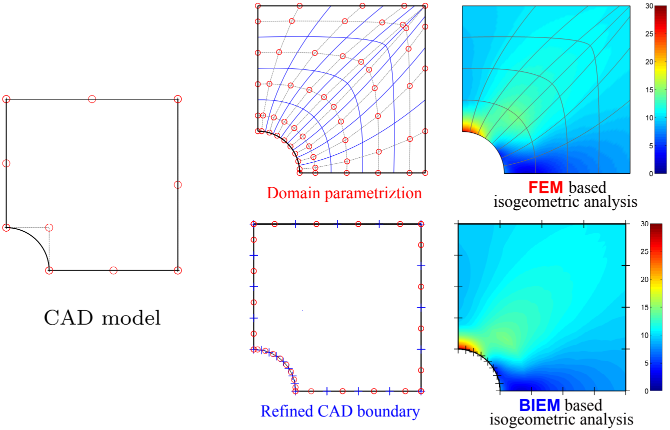
Isogeometric Analysis and Shape Optimization via Boundary Integral

Kang Li Xiaoping Qian

ILLINOIS INSTITUTE OF TECHNOLOGY



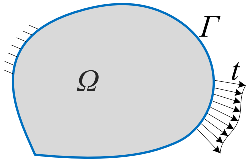
Isogeometric analysis



- FEM based isogeometric analysis (Hughes et al 2005)
  - Advantages: exact geometry, nodal efficiency etc.
  - Domain parametrization (challenging)
- BIEM based isogeometric analysis
  - Boundary parametrization (directly available from CAD)

Outline

Linear elasticity by the BIEM



Linear elasticity

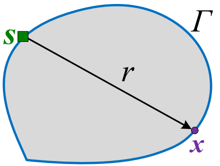
Strong form

$$\begin{cases} \sigma_{ij,j} + b_i = 0 & \text{Equilibrium (in } \Omega) \\ u_i = \bar{u}_i & \text{Dirichlet BC (on } \Gamma_D) \\ t_i = \sigma_{ij} n_j = \bar{t}_i & \text{Neumann BC (on } \Gamma_N) \end{cases}$$

where: $\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$ Constitutive

$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$ Kinematics

■ Source point ● Field point



Boundary Integral Equation (BIE) form

$$C_{ij}(\mathbf{s}) u_j(\mathbf{s}) = \int_\Gamma \left[U^*_{ij}(\mathbf{s}, \mathbf{x}) t_j(\mathbf{x}) - T^*_{ij}(\mathbf{s}, \mathbf{x}) u_j(\mathbf{x}) \right] d\Gamma(\mathbf{x})$$

- *Betti's theorem*: $W(\mathbf{t}, \mathbf{U}^*) = W(\mathbf{T}^*, \mathbf{u})$
- **Fundamental solution** U^*_{ij}, T^*_{ij}
  - Function of r, where $r = |\mathbf{r}| = |\mathbf{x} - \mathbf{s}|$
  - Physical meaning: unit point load response

Discrete BIE for solving

- Discrete BIE after boundary element discretization

$$\mathbf{C}(\mathbf{s})\mathbf{u}(\mathbf{s}) + \sum_{l=1}^{n_{el}} \left[\int_{\Gamma_l} \mathbf{T}^*(\mathbf{s},\mathbf{x})\mathbf{N}_l \, d\Gamma(\mathbf{x}) \right] \mathbf{u}_l = \sum_{l=1}^{n_{el}} \left[\int_{\Gamma_l} \mathbf{U}^*(\mathbf{s},\mathbf{x})\mathbf{N}_l \, d\Gamma(\mathbf{x}) \right] \mathbf{t}_l$$

- Shape function matrix (n_{local}: element node/control point number)

$$\mathbf{N}_l = \begin{bmatrix} N_1\mathbf{I}_d & N_2\mathbf{I}_d & \dots & N_{n_{local}}\mathbf{I}_d \end{bmatrix}$$

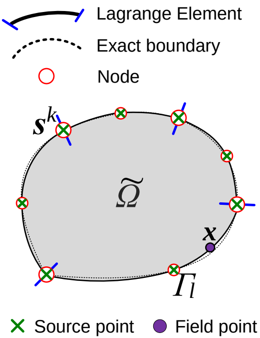
- Collocating n_{global} source points (a.k.a **collocation points**) and applying discrete BIE

$$[\mathbf{H}]\{\mathbf{u}\} = [\mathbf{G}]\{\mathbf{t}\}$$

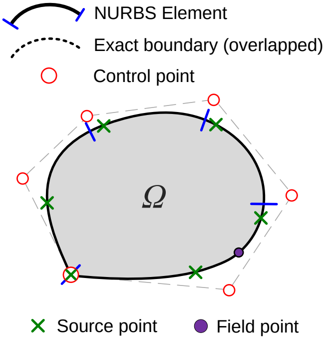
- Separation of knowns and unknowns

$$[\mathbf{A}]\{\mathbf{z}\} = [\mathbf{f}]$$

Collocation: NURBS vs. Lagrange



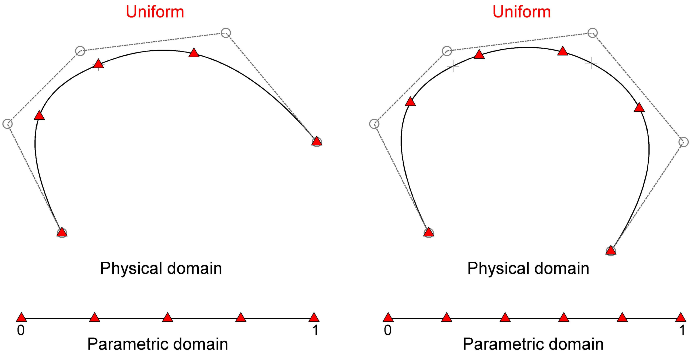
Lagrange elements



NURBS elements

- Collocation points must lie on the boundary
- Lagrange BIEM: collocate source points at nodes
- NURBS BIEM: where to collocate?

Collocation schemes for NURBS



$n_{cp}(=5)$ control points, cubic
knots: $\{0,0,0,0,\frac{1}{2},1,1,1,1\}$

$n_{cp}(=6)$ control points, cubic
knots: $\{0,0,0,0,\frac{1}{3},\frac{2}{3},1,1,1,1\}$

- Consider 4 collocation schemes

  Uniform distribution Gaussian quadrature Maximum basis Greville abscissae

  - 2 end points chosen as collocation points
  - Internal $(n_{cp}-2)$ collocation points in general different

Singularity in the BIEM

- Occurs in fundamental solution as $r = |\mathbf{x} - \mathbf{s}| \to 0$

2D: $$U_{ij}^* = \frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{ij}\ln\frac{1}{r} + r_{,i}\,r_{,j}\right]$$

$$T_{ij}^* = -\frac{1}{4\pi\mu(1-\nu)}\frac{1}{r}\left\{\frac{\partial r}{\partial \mathbf{n}}[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}] + (1-2\nu)(n_i r_{,j} - n_j r_{,i})\right\}$$

3D: $$U_{ij}^* = \frac{1}{16\pi\mu(1-\nu)}\frac{1}{r}\left[(3-4\nu)\delta_{ij} + r_{,i}\,r_{,j}\right]$$

$$T_{ij}^* = -\frac{1}{8\pi\mu(1-\nu)}\frac{1}{r^2}\left\{\frac{\partial r}{\partial \mathbf{n}}[(1-2\nu)\delta_{ij} + 3r_{,i}r_{,j}] + (1-2\nu)(n_i r_{,j} - n_j r_{,i})\right\}$$

- Comes in two types

Singularity	Integrand	Anti-derivative
Weak	singular	non-singular
Strong	singular	singular

- Influences analysis accuracy significantly

Strong singularity (Rigid Body Motion)

- Pure translation results in no shape change: $\{\mathbf{t}\}_i = 0$

$$[\hat{\mathbf{C}}]\,\{\mathbf{u}\} + [\hat{\mathbf{H}}]\,\{\mathbf{u}\} = \mathbf{O} \implies [\hat{\mathbf{C}}] + [\hat{\mathbf{H}}] = \mathbf{O}$$

  - Modify entries in $[\hat{\mathbf{H}}]$ for the k-th collocation point $\mathbf{s}^k$ ($k = 1, ..., n_{global}$)

- Lagrange elements

$$\mathbf{H}_{k,k} = \hat{\mathbf{H}}_{k,k} + \hat{\mathbf{C}}_{k,k} = -\sum_{j=1}^{n_{global}} \hat{\mathbf{H}}_{k,j}\,(j \neq k)$$

- NURBS elements

$$\mathbf{H}_{\chi_i^k,k} = \hat{\mathbf{H}}_{\chi_i^k,k} + N_{\chi_i^k}(\zeta^k)\,\hat{\mathbf{C}}_{k,k} \quad (i = 1, ..., n_{local})$$

  - ζ^k: parameter corresponding to $\mathbf{s}^k$
  - χ^k: index set of the influential control points of $\mathbf{s}^k$

2D Weak singularity (Separation by change of variable)

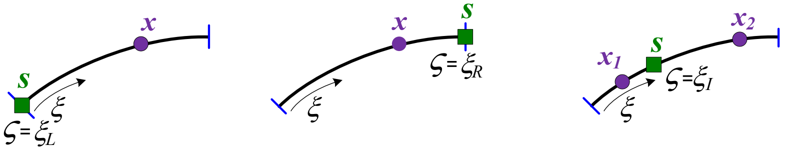
- Change of variable $\eta = g(\xi)$

$$f(\xi)\ln\frac{1}{r(\xi)} = f[g^{-1}(\eta)]\ln\frac{1}{\eta\psi(\eta)} = \underbrace{(f\circ g^{-1})(\eta)\ln\frac{1}{\eta}}_{\text{Logarithmic}} + \underbrace{f(\xi)\ln\frac{1}{(\psi\circ g)(\xi)}}_{\text{Non-logarithmic}}$$

- Separation of singular part in r

$$r = |\mathbf{x}-\mathbf{s}| = (\xi-\zeta)\,\frac{r}{(\xi-\zeta)}$$

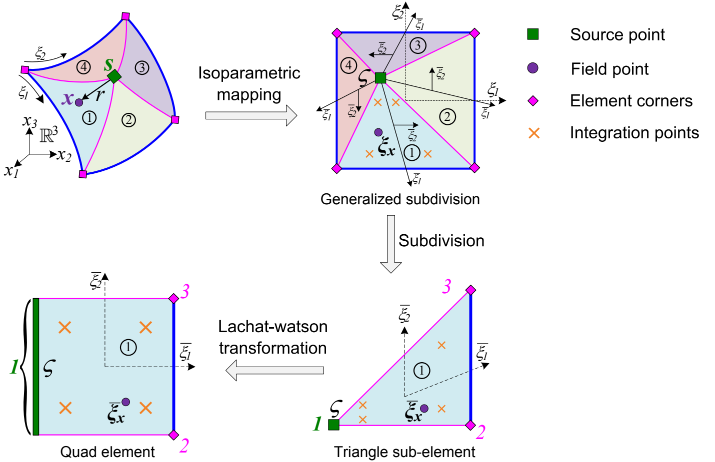
- Generalized form of $\eta = g(\xi)$



Left-end collocation — Right-end collocation — Mid-side collocation

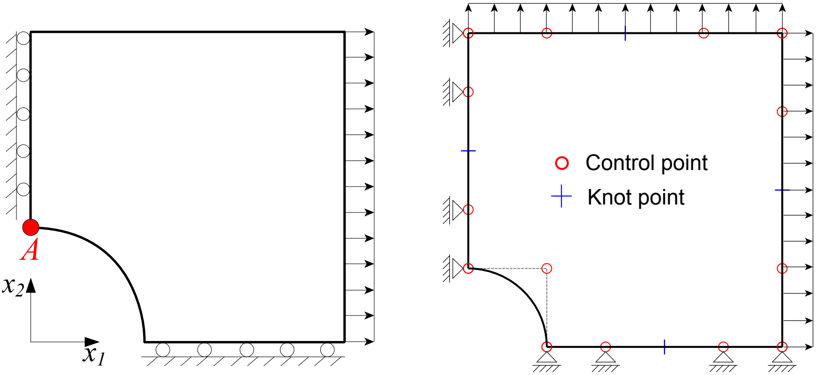
$$\eta_{\mathbf{x}} = \frac{\xi-\xi_L}{\xi_R-\xi_L} \qquad \eta_{\mathbf{x}} = \frac{\xi_R-\xi}{\xi_R-\xi_L} \qquad \eta_{\mathbf{x}_1} = \frac{\xi-\xi_L}{\xi_I-\xi_L},\ \eta_{\mathbf{x}_2} = \frac{\xi_R-\xi}{\xi_R-\xi_I}$$

3D Weak singularity (Lachat-Watson Transformation)



$$\int_{\Gamma_l} \mathbf{U}_l^* \mathbf{N}_l \, d\Gamma = \int_{-1}^{1}\int_{-1}^{1} \mathbf{U}_l^* \mathbf{N}_l J \, d\xi_1 d\xi_2 = \sum_{i=1}^{4} \int_{-1}^{1}\int_{-1}^{1} \mathbf{U}_l^* \mathbf{N}_l J \, \overline{J}_i \, d\overline{\xi}_1 d\overline{\xi}_2$$

Benchmark problem: plate hole

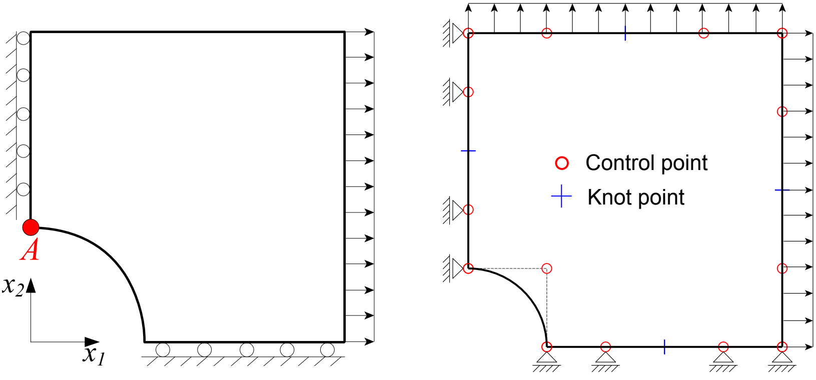


Infinite quarter plate

Finite plate with known BC

- Analytical solution for infinite problem as boundary condition
- Physical quantities measured for convergence study
  - Local stress $\sigma_{11}(A)$
  - Strain energy

Benchmark problem: plate hole

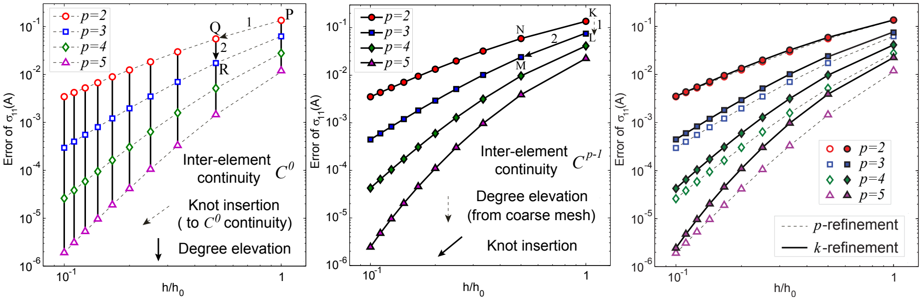


Infinite quarter plate

Finite plate with known BC

- Analytical solution for infinite problem as boundary condition
- Physical quantities measured for convergence study
  - Local stress $\sigma_{11}(A)$
  - Strain energy

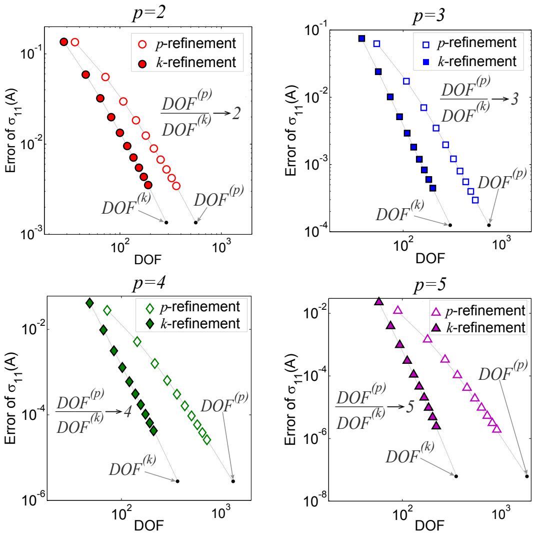
p- vs. k-refinement: per element



p-refinement k-refinement Overlay

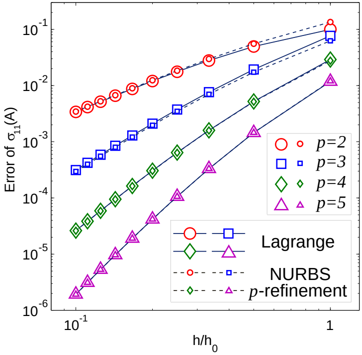
- NURBS element refinement
  - ▶ p-refine: ① Knot insertion (until C^0) $\rightarrow$ ② Degree elevation
  - ▶ k-refine: ① Degree elevation (from coarsest) $\rightarrow$ ② Knot insertion
- Convergence per element: p-refinement $\approx$ k-refinement

p- vs. k-refinement: per DOF

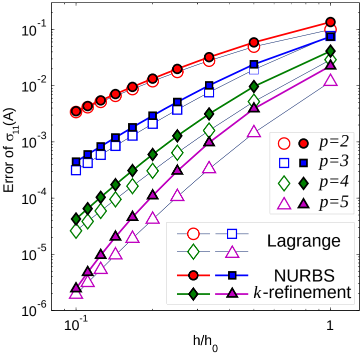


- k-refinement: approaching p times more nodal efficient

Lagrange vs. NURBS BIEM: per element



Lagrange vs. NURBS p-refinement

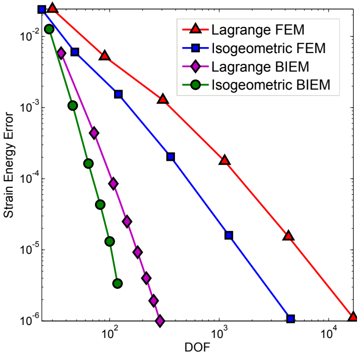


Lagrange vs. NURBS k-refinement

- Convergence per element

  NURBS k-refinement $\approx$ NURBS p-refinement $\approx$ Lagrange
- NURBS k-refinement: similar nodal advantage over Lagrange BIEM

Lagrange vs. NURBS & volumetric vs. boundary: per DOF

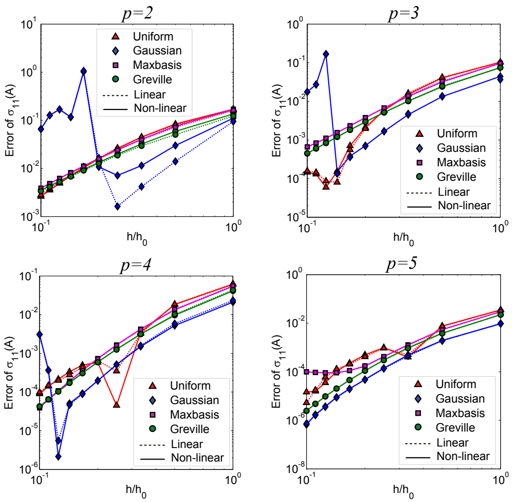


Lagrange vs. NURBS, FEM vs. BIEM

- NURBS (k-refinement) is more nodal efficient than Lagrange
- BIEM is more nodal efficient than FEM
- Nodal efficiency

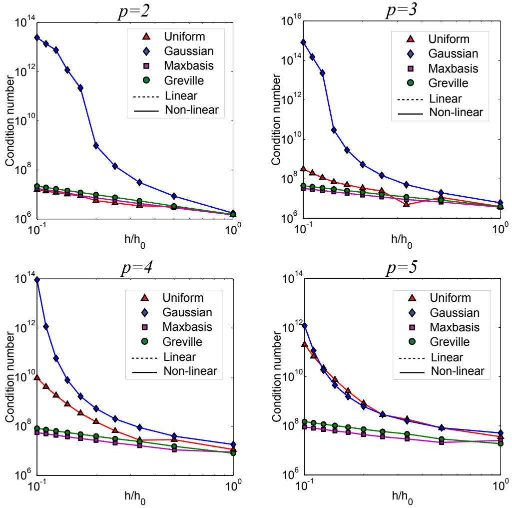
  Isogeometric BIEM $>$ Lagrange BIEM $>$Isogeometric FEM $>$ Lagrange FEM

Collocation schemes & solution space linearity



- Accuracy (collocation): Greville, Maxbasis $>$ Uniform, Gaussian
- Accuracy (solution space linearity): non-linear $\approx$ linear

Condition number



- Stability: Maxbasis, Greville $>$ Uniform, Gaussian
- Greville abscissae has better overall accuracy and stability

BIEM-based shape optimization

Structural shape optimization via BIEM

$$
(\mathbb{SO})_{\text{nf}} \begin{cases} \min\limits_{\boldsymbol{\alpha}} & \hat{f}(\boldsymbol{\alpha}, \mathbf{u}(\boldsymbol{\alpha}), \mathbf{t}(\boldsymbol{\alpha})) \\ s.t. & \hat{h}_i(\boldsymbol{\alpha}, \mathbf{u}(\boldsymbol{\alpha}), \mathbf{t}(\boldsymbol{\alpha})) = 0 \\ & \hat{g}_j(\boldsymbol{\alpha}, \mathbf{u}(\boldsymbol{\alpha}), \mathbf{t}(\boldsymbol{\alpha})) \leqslant 0 \end{cases}
$$

- State variables found by equilibrium equation

$$
[\mathbf{H}(\boldsymbol{\alpha})]\,\{\mathbf{u}(\boldsymbol{\alpha})\} \;=\; [\mathbf{G}(\boldsymbol{\alpha})]\,\{\mathbf{t}(\boldsymbol{\alpha})\}
$$

- Analytical sensitivities for gradient-based optimization

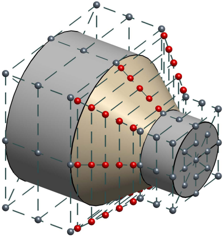
$$
\frac{\partial \hat{f}}{\partial \alpha_k} = \frac{\partial f}{\partial \alpha_k} + \left(\frac{\partial f}{\partial \mathbf{u}}\right)^T \frac{\partial \mathbf{u}}{\partial \alpha_k} + \left(\frac{\partial f}{\partial \mathbf{t}}\right)^T \frac{\partial \mathbf{t}}{\partial \alpha_k}
$$

- Differentiation of discrete BIE

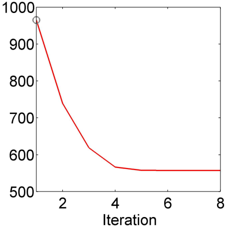
$$
[\mathbf{H}']\{\mathbf{u}\} + [\mathbf{H}]\{\mathbf{u}'\} = [\mathbf{G}']\{\mathbf{t}\} + [\mathbf{G}]\{\mathbf{t}'\}
$$

Example 1: fillet profile

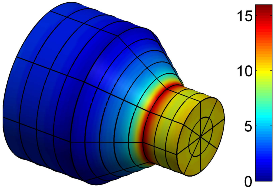
Design NURBS (Iter.=1)



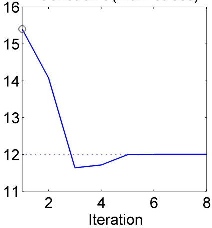
Objective (Volume)



Stress field on NURBS elements (Iter.=1)



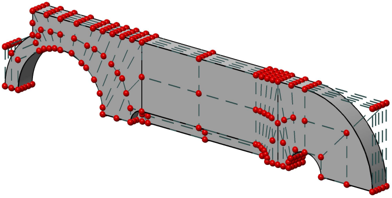
Constraint (Max. stress)



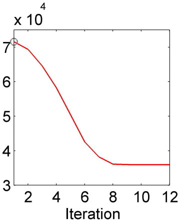
$(\mathbb{SO})$: ▶ Minimize volume ▶ s.t. $\sigma \leqslant \sigma^*$

Example 2: connecting rod

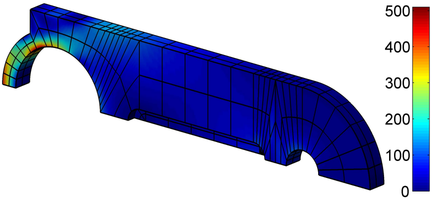
Design NURBS (Iter.=01)



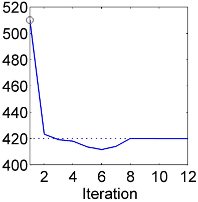
Objective (Volume)



Stress field on NURBS elements (Iter.=01)



Constraint (Max. stress)



$(\mathbb{SO})$: ▶ Minimize volume ▶ s.t. $\sigma \leqslant \sigma^*$

Conclusions and future work

- A **boundary integral** based isogeometric analysis method
  - **Greville abscissae** collocation for accurate and stable analysis
  - **Per-DOF superiority** of NURBS BIEM over Lagrange BIEM
  - **3D shape optimization**
- Future work
  - Mathematical proof of collocation scheme performance
  - Extension to trimmed NURBS