Isogeometric Analysis Tutorial

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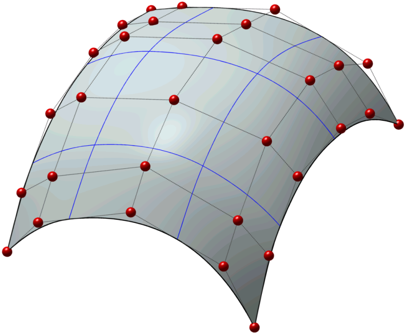
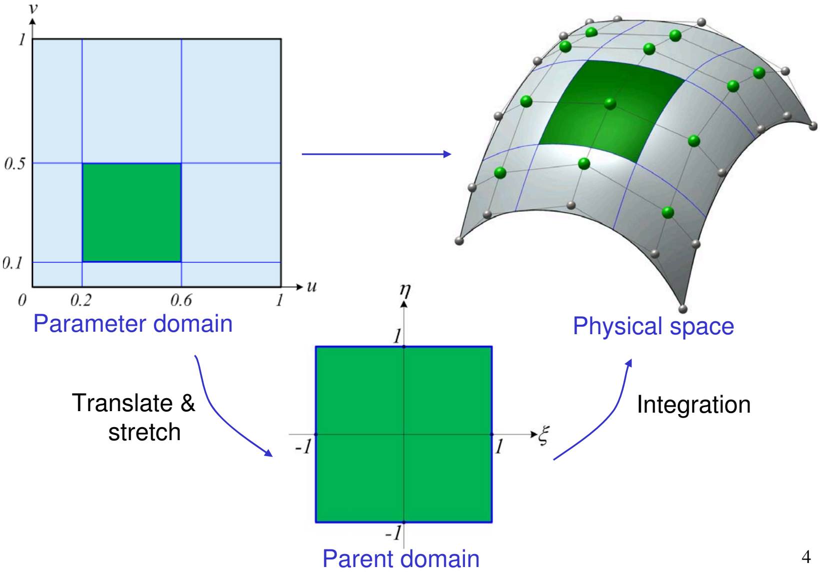
NURBS geometry



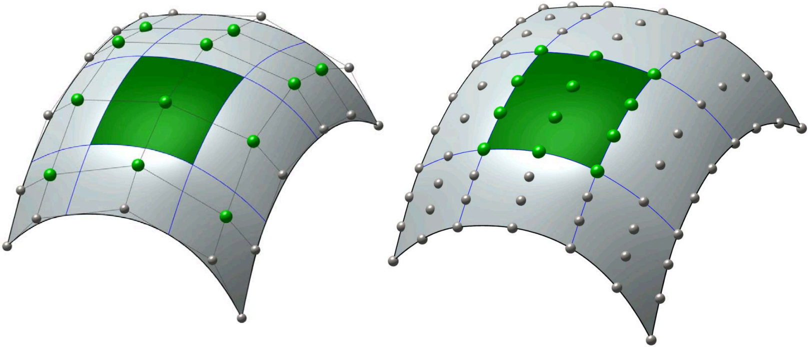
- Degrees: $p = 2$, $q = 3$
- Knots:
  $\boldsymbol{\xi} = \{0, 0, 0, 0.2, 0.6, 1, 1, 1\}$,
  $\boldsymbol{\eta} = \{0, 0, 0, 0, 0.1, 0.5, 1, 1, 1, 1\}$
- Control points: 5×6 $\mathbf{P}_{i,j}$

$$\mathbf{S}(\xi, \eta) = \frac{\sum_{i=0,j=0}^{m,n} N_{i,p} N_{j,q} w_{i,j} \mathbf{P}_{i,j}}{\sum_{i=0,j=0}^{m,n} N_{i,p} N_{j,q} w_{i,j}}$$

NURBS based isogeometric analysis

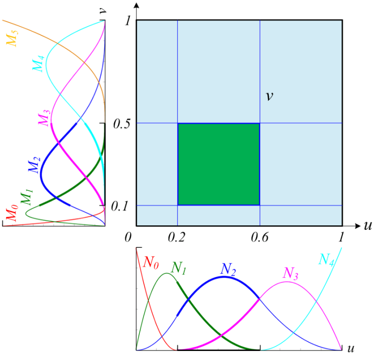


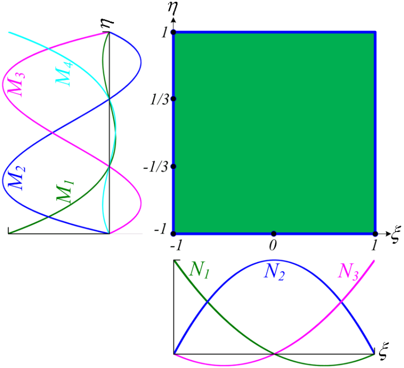
NURBS versus Lagrange: elements and nodes



Same number of elements, NURBS use fewer nodes

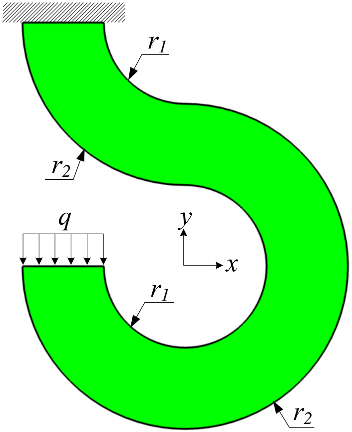
NURBS versus Lagrange: shape functions





- NURBS: C^{p-m} inter-element continuity, partition of unity, non-negative, $(p+1)$ spans, non-interpolatory
- Lagrange: Kronecker delta, interpolatory

Illustrative Example: 2D Hook



- Plain stress problem
- Geometry

  $$r_1 = 1,\ r_2 = 2$$

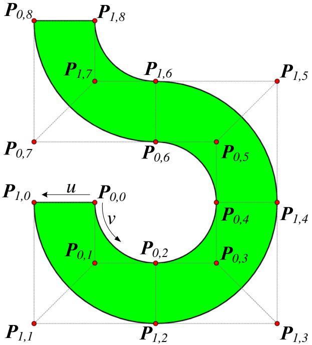
- Boundary conditions
  - Dirichlet BC (fixed at top)

    $u_x = 0, u_y = 0$
  - Neumann BC

    $q = 1$
- Material property constants

  $$E = 2 \times 10^{11},\ \nu = 0.3$$

NURBS Geometry



NURBS Control Net

Index *i*	Index *j*	$P_{i,j}$	$w_{i,j}$	Index *i*	Index *j*	$P_{i,j}$	$w_{i,j}$
0	0	(-1,0)	1	1	4	(2,0)	1
1	0	(-2,0)	1	0	5	(1,1)	0.7071
0	1	(-1,-1)	0.7071	1	5	(2,2)	0.7071
1	1	(-2,-2)	0.7071	0	6	(0,1)	1
0	2	(0,-1)	1	1	6	(0,2)	1
1	2	(0,-2)	1	0	7	(-2,1)	0.7071
0	3	(1,-1)	0.7071	1	7	(-1,2)	0.7071
1	3	(2,-2)	0.7071	0	8	(-2,3)	1
0	4	(1,0)	1	1	8	(-1,3)	1

Control Point Table

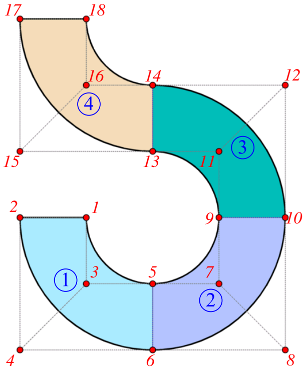
- Degree: $p = 1, q = 2$
- Control point (*hereafter called "CP"*) number: $n_{cp} = 2 \times 9 = 18 \, (m = 1, n = 8)$
- Knot vector: $\boldsymbol{\xi} = \{0, 0, 1, 1\}$

$$\boldsymbol{\eta} = \{0, 0, 0, 0.25, 0.25, 0.5, 0.5, 0.75, 0.75, 1, 1, 1\}$$

- NURBS Representation

$$\mathbf{S}(u, v) = \sum_{i=0, j=0}^{m,n} R_{i,j} \mathbf{P}_{i,j} \quad \text{where:} \quad R_{i,j} = \frac{N_{i,p} M_{j,q} w_{i,j}}{\sum_{\hat{i}=0, \hat{j}=0}^{m,n} N_{\hat{i},p} M_{\hat{j},q} w_{\hat{i},\hat{j}}}$$

NURBS Element



NURBS Elements

k	Index i	Index j	k	Index i	Index j
1	0	0	10	1	4
2	1	0	11	0	5
3	0	1	12	1	5
4	1	1	13	0	6
5	0	2	14	1	6
6	1	2	15	0	7
7	0	3	16	1	7
8	1	3	17	0	8
9	0	4	18	1	8

CP Numbering

Element	Element Control Points	Knot Span u	Knot Span v
1	1,2;3,4;5,6	[0 1]	[0 0.25]
2	5,6;7,8;9,10	[0 1]	[0.25 0.5]
3	9,10;11,12;13,14	[0 1]	[0.5 0.75]
4	13,14;15,16;17,18	[0 1]	[0.75 1]

Element	Local Knot Vector u	Local Knot Vector v
1	0, 0, 1, 1	0, 0, 0, 0.25, 0.25, 0.5
2	0, 0, 1, 1	0, 0.25, 0.25, 0.5, 0.5, 0.75
3	0, 0, 1, 1	0.25, 0.5, 0.5, 0.75, 0.75, 1
4	0, 0, 1, 1	0.5, 0.75, 0.75, 1, 1, 1

Element Table

- Element number: $n_{el} = 4$
- For numbering convenience, let: $k = (i+1) + (m+1)j$
- Rewrite NURBS representation

$$\mathbf{S}(u,v) = \sum_{k=1}^{n_{cp}} R_k \mathbf{P}_k \quad \text{where: } R_k = R_{i,j}, \mathbf{P}_k = \mathbf{P}_{i,j}$$

A Typical Isogeometric Analysis Framework

- Solve *Global Stiffness Equations*

$$\mathbf{K}_{(36\times36)}\,\mathbf{d}_{(36\times1)} = \mathbf{f}_{(36\times1)}$$

  to find: $\mathbf{d} = \begin{bmatrix} u_{x_1} & u_{y_1} & u_{x_2} & u_{y_2} & \ldots & u_{x_{18}} & u_{y_{18}} \end{bmatrix}^T$

- *Global Stiffness Matrix* and *Global Force Vector*

$$\mathbf{K} = \sum_{e=1}^{n_{el}} \mathbf{L}_e^T \Big(\int_{\Omega_e} \mathbf{B}_e^T \mathbf{D}_e \mathbf{B}_e d\Gamma \Big) \mathbf{L}_e = \sum_{e=1}^{n_{el}} \mathbf{L}_e^T \Big(\int_{-1}^{1} \int_{-1}^{1} \mathbf{B}_e^T \mathbf{D}_e \mathbf{B}_e |\mathbf{J}_\Omega| \, d\xi d\eta \Big) \mathbf{L}_e$$

$$\mathbf{f} = \sum_{e=1}^{n_{el}} \mathbf{L}_e^T \Big(\int_{\Gamma_e} \mathbf{N}_e^T \mathbf{t} d\Gamma \Big) \mathbf{L}_e = \sum_{e=1}^{n_{el}} \mathbf{L}_e^T \Big(\int_{-1}^{1} \mathbf{N}_e^T \mathbf{t} |\mathbf{J}_\Gamma| d\xi \Big) \mathbf{L}_e$$

- Computation dependence

  $\mathbf{B}_e \Rightarrow R_{k,x}, R_{k,y} \Rightarrow R_{k,\xi}, R_{k,\eta} \Rightarrow N'(u), M'(v), \frac{du}{d\xi}, \frac{du}{d\xi}$

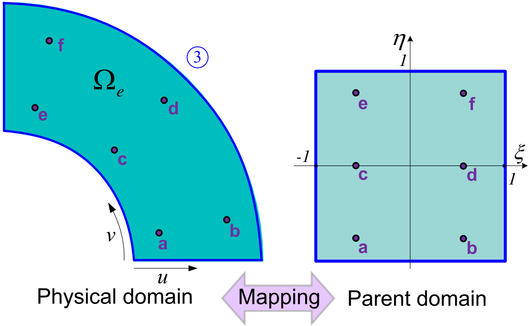
  $\mathbf{J}_\Omega \Rightarrow x_{,\xi}, y_{,\xi}, x_{,\eta}, y_{,\eta} \Rightarrow R_{k,\xi}, R_{k,\eta} \Rightarrow N'(u), M'(v), \frac{du}{d\xi}, \frac{du}{d\xi}$

  $\mathbf{N}_e \Rightarrow R_k \Rightarrow N(u), M(v)$

  $\mathbf{J}_\Gamma \Rightarrow x_{,\xi}, y_{,\xi} \Rightarrow R_{k,\xi} \Rightarrow N'(u), \frac{du}{d\xi}, \frac{du}{d\xi}$

- Therefore, calculating $\mathbf{K}$ and $\mathbf{f}$ in global stiffness equation boils down to evaluating NURBS basis functions and their derivatives

Guass Quadrature for Element Matrix Calculation



Gauss Point	Index I	Abscissa ξ_I	Abscissa η_I	Weight β_I
a	1	-0.5774	-0.7746	0.5556
b	2	0.5774	-0.7746	0.5556
c	3	-0.5774	0	0.8889
d	4	0.5774	0	0.8889
e	5	-0.5774	0.7746	0.5556
f	6	0.5774	0.7746	0.5556

Gauss Point Table

- Take element ③ for example
- Element matrix (Linear $\times$ Quadratic $\Rightarrow 2 \times 3$ Gauss points)

$$\mathbf{K}_e = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}_e^T \mathbf{D}_e \mathbf{B}_e |\mathbf{J}|\, d\xi d\eta = \sum_{I=1}^{6} \beta_I f(\xi_I, \eta_I)$$

where: $\quad f(\xi_I, \eta_I) = \mathbf{B}_e^T(\xi_I, \eta_I)\mathbf{D}_e\mathbf{B}_e(\xi_I, \eta_I)|\mathbf{J}(\xi_I, \eta_I)|$

Element Quantities

- Element displacement vector

$$\mathbf{d}_{e\ (12\times1)} = \begin{bmatrix} u_{x_9} & u_{y_9} & u_{x_{10}} & u_{y_{10}} & \dots & u_{x_{14}} & u_{y_{14}} \end{bmatrix}^T$$

- Element CP coordinate matrix (k-th CP is $\mathbf{P}_k = [x_k, y_k]$)

$$\mathbf{C}_{e\ (6\times2)} = \begin{bmatrix} x_9 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} \\ y_9 & y_{10} & y_{11} & y_{12} & y_{13} & y_{14} \end{bmatrix}^T$$

- N-matrix

$$\mathbf{N}_{e\ (2\times12)} = \begin{bmatrix} R_9 & 0 & R_{10} & 0 & \dots & R_{14} & 0 \\ 0 & R_9 & 0 & R_{10} & \dots & 0 & R_{14} \end{bmatrix}$$

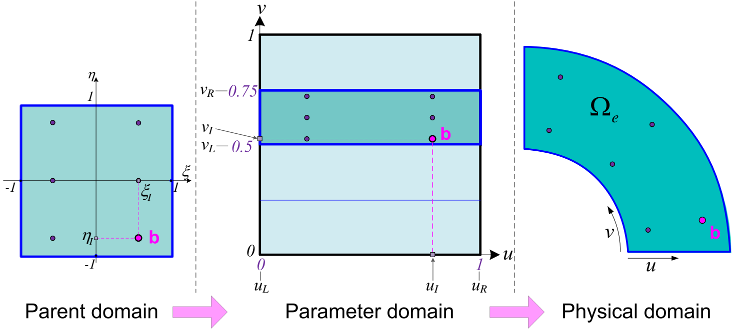
- B-matrix (“,” denotes differentiation)

$$\mathbf{B}_{e\ (3\times12)} = \begin{bmatrix} R_{9,x} & 0 & R_{10,x} & 0 & \dots & R_{14,x} & 0 \\ 0 & R_{9,y} & 0 & R_{10,y} & \dots & 0 & R_{14,y} \\ R_{9,y} & R_{9,x} & R_{10,y} & R_{10,x} & \dots & R_{14,y} & R_{14,x} \end{bmatrix}$$

Related Identities

- Define *Element shape function matrix*
$$\mathbf{R}_e = \begin{bmatrix} R_9 & R_{10} & R_{11} & R_{12} & R_{13} & R_{14} \end{bmatrix}$$
where R_k is the shape function of the k-th CP $\mathbf{P}_k$
- A point in the element is
$$\mathbf{S}(u,v) = \begin{bmatrix} x & y \end{bmatrix} = \mathbf{R}_e \mathbf{C}_e$$
- Differentiate w.r.t ξ and η to find Jacobian
$$\mathbf{J} = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{,\xi} \\ \mathbf{S}_{,\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{e,\xi} \\ \mathbf{R}_{e,\eta} \end{bmatrix} \mathbf{C}_e$$
- Notice by chain rule ($l = 9, 10, 11, 12, 13, 14$)
$$R_{l,\xi} = R_{l,x} x_{,\xi} + R_{l,y} y_{,\xi}$$
$$R_{l,\eta} = R_{l,x} x_{,\eta} + R_{l,y} y_{,\eta}$$
- Write in matrix form
$$\begin{bmatrix} R_{l,\xi} \\ R_{l,\eta} \end{bmatrix} = \mathbf{J} \begin{bmatrix} R_{l,x} \\ R_{l,y} \end{bmatrix} \qquad \text{Invert:} \quad \begin{bmatrix} R_{l,x} \\ R_{l,y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} R_{l,\xi} \\ R_{l,\eta} \end{bmatrix}$$

Gauss Point Mapping



- Illustrative Gauss point **b**: $\xi_I = 0.5774, \eta_I = -0.7746, \beta_I = 0.5556$
- Look up knot span for element ③ in element data table

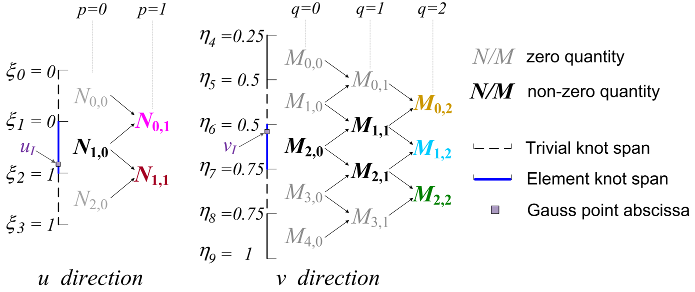
$$u_L = 0, u_R = 1 \qquad v_L = 0.5, v_R = 0.75$$

- Mapping from $Parent$ to $Parameter$ domain

$$u = \frac{1}{2}(1-\xi)u_L + \frac{1}{2}(1+\xi)u_R \qquad v = \frac{1}{2}(1-\eta)v_L + \frac{1}{2}(1+\eta)v_R$$

- Parameters after mapping are: $u_I = 0.7887,\ v_I = 0.5282$

Basis Function in Each Direction



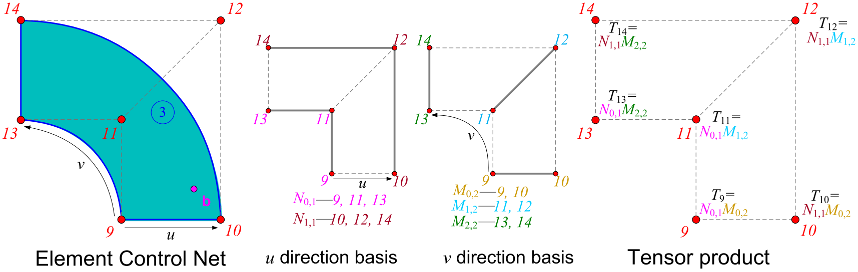
- Global knot vector
  - u direction: $\boldsymbol{\xi}_{(1\times4)} = \{\xi_0, \xi_1, \xi_2, \xi_3\}$
  - v direction: $\boldsymbol{\eta}_{(1\times12)} = \{\eta_0, \eta_1, \eta_2, ..., \eta_{11}\}$
- Local knot vector for element ③
  - u direction: $\{\xi_0, \xi_1, \xi_2, \xi_3\}$
  - v direction: $\{\eta_4, \eta_5, \eta_6, \eta_7, \eta_8, \eta_9\}$
- 2 basis functions in u direction ($p = 1$)

$$N_{0,1}(u_I) = 0.2113, \quad N_{1,1}(u_I) = 0.7887$$

- 3 basis functions in v direction ($q = 2$)

$$M_{0,2}(v_I) = 0.7873, \quad M_{1,2}(v_I) = 0.2000, \quad M_{2,2}(v_I) = 0.0127$$

Tensor Product of Basis Function



- Basis tensor product of the k-th CP

$$\mathbf{T}_k = N_{i,p}M_{j,q} = N_{i,1}M_{j,2}$$

- Basis tensor product vector

$$\mathbf{T}_e = \begin{bmatrix} T_9 & T_{10} & T_{11} & T_{12} & T_{13} & T_{14} \end{bmatrix}$$
$$= \begin{bmatrix} 0.1664 & 0.6209 & 0.0423 & 0.1577 & 0.0027 & 0.0100 \end{bmatrix}$$

Shape Function

- Define *Element CP weight vector*

$$\mathbf{w}_e = \begin{bmatrix} w_9 & w_{10} & w_{11} & w_{12} & w_{13} & w_{14} \end{bmatrix}^T$$
$$= \begin{bmatrix} 1 & 1 & 0.7071 & 0.7071 & 1 & 1 \end{bmatrix}^T$$

- Shape function of the k-th CP could be expressed as

$$R_k = \frac{N_{i,p} M_{j,q} w_{i,j}}{\sum_{\hat{i}=0,\hat{j}=0}^{m,n} N_{\hat{i},p} N_{\hat{j},q} w_{\hat{i},\hat{j}}} = \frac{T_k w_{i,j}}{\sum_{\hat{k}=0}^{6} T_{\hat{k}} w_{\hat{i},\hat{j}}} = \frac{T_k w_{i,j}}{\mathbf{T}_e \mathbf{w}_e}$$

- Element shape function vector is then

$$\mathbf{R}_e = \begin{bmatrix} R_9 & R_{10} & R_{11} & R_{12} & R_{13} & R_{14} \end{bmatrix}$$
$$= \begin{bmatrix} 0.0943 & 0.7420 & 0.0169 & 0.1333 & 0.0015 & 0.0120 \end{bmatrix}$$

Tensor Product of Basis Function Derivative

- 2 basis derivatives in u direction

$$(N_{0,1})_{,u}(u_I) = -1, (N_{1,1})_{,u}(u_I) = 1$$

- 3 basis derivatives in v direction

$$(M_{0,2})_{,v}(v_I) = -7.0984, (M_{1,2})_{,v}(v_I) = 6.1968, (M_{2,2})_{,v}(v_I) = 0.9016$$

- Basis derivative tensor product of the k-th CP

$$(\mathbf{T}_k)_{,u} = (N_{i,1})_{,u} \cdot M_{j,2} \qquad (\mathbf{T}_k)_{,v} = N_{i,1} \cdot (M_{j,2})_{,v}$$

- Basis derivative tensor product vector

$$\begin{aligned}(\mathbf{T}_e)_{,u} &= \begin{bmatrix}(T_9)_{,u} & (T_{10})_{,u} & (T_{11})_{,u} & (T_{12})_{,u} & (T_{13})_{,u} & (T_{14})_{,u}\end{bmatrix} \\ &= \begin{bmatrix}-0.7873 & 0.7873 & -0.2000 & 0.2000 & -0.0127 & 0.0127\end{bmatrix}\end{aligned}$$

$$\begin{aligned}(\mathbf{T}_e)_{,v} &= \begin{bmatrix}(T_9)_{,v} & (T_{10})_{,v} & (T_{11})_{,v} & (T_{12})_{,v} & (T_{13})_{,v} & (T_{14})_{,v}\end{bmatrix} \\ &= \begin{bmatrix}-1.5001 & -5.5983 & 1.3095 & 4.8872 & 0.1905 & 0.7111\end{bmatrix}\end{aligned}$$

Shape Function Derivative

- Differentiate shape functions w.r.t ξ and η

$$R_{k,\xi} = \frac{(T_k)_{,u} w_{i,j} \cdot \mathbf{T}_e \mathbf{w}_e - T_k w_{i,j} \cdot (\mathbf{T}_e)_{,u} \mathbf{w}_e}{(\mathbf{T}_e \mathbf{w}_e)^2} \cdot \frac{du}{d\xi}$$

$$R_{k,\eta} = \frac{(T_k)_{,v} w_{i,j} \cdot \mathbf{T}_e \mathbf{w}_e - T_k w_{i,j} \cdot (\mathbf{T}_e)_{,v} \mathbf{w}_e}{(\mathbf{T}_e \mathbf{w}_e)^2} \cdot \frac{dv}{d\eta}$$

- Domain mapping gives

$$\frac{du}{d\xi} = \frac{1}{2}(u_R - u_L) \qquad \frac{dv}{d\eta} = \frac{1}{2}(v_R - v_L)$$

- Shape function derivative vector

$$\begin{aligned} \mathbf{R}_{e,\xi} &= \begin{bmatrix} (R_9)_{,\xi} & (R_{10})_{,\xi} & (R_{11})_{,\xi} & (R_{12})_{,\xi} & (R_{13})_{,\xi} & (R_{14})_{,\xi} \end{bmatrix} \\ &= \begin{bmatrix} -0.4181 & 0.4181 & -0.0751 & 0.0751 & -0.0067 & 0.0067 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{R}_{e,\eta} &= \begin{bmatrix} (R_9)_{,\eta} & (R_{10})_{,\eta} & (R_{11})_{,\eta} & (R_{12})_{,\eta} & (R_{13})_{,\eta} & (R_{14})_{,\eta} \end{bmatrix} \\ &= \begin{bmatrix} -0.1566 & -0.5844 & 0.1306 & 0.4874 & 0.0260 & 0.0970 \end{bmatrix} \end{aligned}$$

B-matrix

- Jacobian

$$\mathbf{J} = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{,\xi} \\ \mathbf{R}_{,\eta} \end{bmatrix}, \ \mathbf{C}_e = \begin{bmatrix} 0.4933 & 0.0819 \\ -0.2199 & 1.3254 \end{bmatrix}$$

- Jacobian inverse and determinant

$$\mathbf{J}^{-1} = \begin{bmatrix} 1.9730 & -0.1219 \\ 0.3274 & 0.7343 \end{bmatrix}, \ |\mathbf{J}| = det(\mathbf{J}) = 0.6717$$

- Isoparametric mapping relation gives

$$\begin{aligned} \mathbf{R}_{e,x} &= \begin{bmatrix} (R_9)_{,x} & (R_{10})_{,x} & (R_{11})_{,x} & (R_{12})_{,x} & (R_{13})_{,x} & (R_{14})_{,x} \end{bmatrix} \\ &= \begin{bmatrix} -0.8059 & 0.8962 & -0.1641 & 0.0888 & -0.0165 & 0.0015 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{R}_{e,y} &= \begin{bmatrix} (R_9)_{,y} & (R_{10})_{,y} & (R_{11})_{,y} & (R_{12})_{,y} & (R_{13})_{,y} & (R_{14})_{,y} \end{bmatrix} \\ &= \begin{bmatrix} -0.2519 & -0.2922 & 0.0713 & 0.3825 & 0.0169 & 0.0734 \end{bmatrix} \end{aligned}$$

- B-matrix: $\mathbf{B}_{e\ (3\times 12)} =$

$$\begin{bmatrix} -0.8059 & 0 & 0.8962 & 0 & -0.1641 & 0 & 0.0888 & 0 & -0.0165 & 0 & 0.0015 & 0 \\ 0 & -0.2159 & 0 & -0.2922 & 0 & 0.0713 & 0 & 0.3825 & 0 & 0.0169 & 0 & 0.0734 \\ -0.2519 & -0.8059 & -0.2922 & 0.8962 & 0.0713 & -0.1641 & 0.3825 & 0.0888 & 0.0169 & -0.0165 & 0.0734 & 0.0015 \end{bmatrix}$$

Element Stiffness Matrix

- Elasticity matrix

$$\mathbf{D}_{e\ (3\times 3)} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix} = \begin{bmatrix} 2.1978 & 0.6593 & 0 \\ 0.6593 & 2.1978 & 0 \\ 0 & 0 & 0.7692 \end{bmatrix} \times 10^{11}$$

- Perform previous steps for all 6 Gauss points and do summation to get element matrix for element ③

$$\mathbf{K}_{e\ (12\times 12)} = \sum_{I=1}^{6} \beta_I \mathbf{B}_e^T \mathbf{D}_e \mathbf{B}_e |\mathbf{J}| =$$

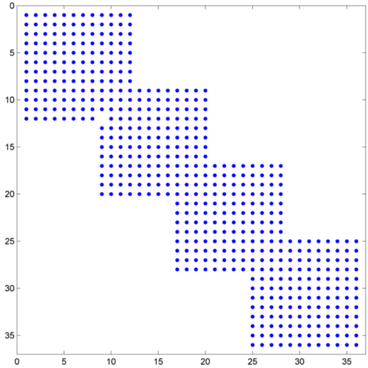
$$\begin{bmatrix}
11.3928 & 3.0408 & -9.8235 & -2.3751 & 2.7592 & 0.6619 & -2.6925 & -2.2283 & -0.3936 & 1.2708 & -1.2423 & -0.3701 \\
3.0408 & 10.6849 & -1.8282 & -3.2906 & 0.3485 & 1.3309 & -2.2254 & -4.6213 & 1.0343 & -0.3936 & -0.3701 & -3.7102 \\
-9.8235 & -1.8282 & 14.4733 & -1.3248 & -5.0512 & 0.0956 & 3.9314 & 1.9220 & -3.7102 & -0.3701 & 0.1803 & 1.5054 \\
-2.3751 & -3.2906 & -1.3248 & 5.8117 & 0.0927 & -2.2626 & 2.2354 & 0.8035 & -0.3701 & -1.2423 & 1.7419 & 0.1803 \\
2.7592 & 0.3485 & -5.0512 & 0.0927 & 5.4872 & 0.3706 & -2.2634 & -1.5693 & 1.3309 & 0.6619 & -2.2626 & 0.0956 \\
0.6619 & 1.3309 & 0.0956 & -2.2626 & 0.3706 & 5.4872 & -1.5693 & -2.2634 & 0.3485 & 2.7592 & 0.0927 & -5.0512 \\
-2.6925 & -2.2254 & 3.9314 & 2.2354 & -2.2634 & -1.5693 & 4.8424 & 1.8655 & -4.6213 & -2.2283 & 0.8035 & 1.9220 \\
-2.2283 & -4.6213 & 1.9220 & 0.8035 & -1.5693 & -2.2634 & 1.8655 & 4.8424 & -2.2254 & -2.6925 & 2.2354 & 3.9314 \\
-0.3936 & 1.0343 & -3.7102 & -0.3701 & 1.3309 & 0.3485 & -4.6213 & -2.2254 & 10.6849 & 3.0408 & -3.2906 & -1.8282 \\
1.2708 & -0.3936 & -0.3701 & -1.2423 & 0.6619 & 2.7592 & -2.2283 & -2.6925 & 3.0408 & 11.3928 & -2.3751 & -9.8235 \\
-1.2423 & -0.3701 & 0.1803 & 1.7419 & -2.2626 & 0.0927 & 0.8035 & 2.2354 & -3.2906 & -2.3751 & 5.8117 & -1.3248 \\
-0.3701 & -3.7102 & 1.5054 & 0.1803 & 0.0956 & -5.0512 & 1.9220 & 3.9314 & -1.8282 & -9.8235 & -1.3248 & 14.4733
\end{bmatrix} \times 10^{10}$$

Global Stiffness Matrix

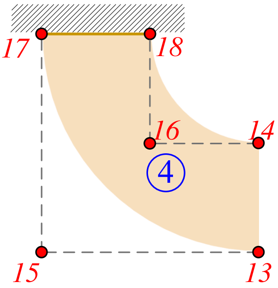
- Perform previous steps for all 4 elements and assemble to obtain ($\mathbf{L}_e$ is element Boolean matrix):

$$\mathbf{K}_{(36\times36)} = \sum_{e=1}^{4} \mathbf{L}_e^T{}_{(36\times12)} \mathbf{K}_{e\,(12\times12)} \mathbf{L}_{e\,(12\times36)}$$

- Global stiffness matrix plot (non-zero entries shown in blue dots)



Dirichlet Boundary Condition



- Entire top boundary of element ④ is fixed

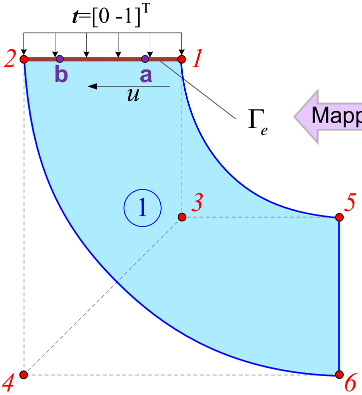
$$u_x = 0 \quad u_y = 0$$

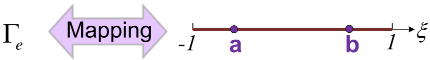
- This Dirichelet BC could be modeled by setting

$$u_{x_{17}} = 0, \; u_{y_{17}} = 0 \qquad u_{x_{18}} = 0, \; u_{y_{18}} = 0$$

- Only need to cross out the $33, 34, 35, 36$-th rows and/or columns in the system of equations

Neumann Boundary Condition





Gauss Point	Index I	Abscissa ξ_I	Weight α_I
a	1	-0.5774	1
b	2	0.5774	1

- Distributed load acts along $\Gamma_e (u \in [0,1], v = 0)$, which traverses the u direction within element ①
- Element force vector (Linear $\Rightarrow$ 2 Gauss points)

$$\mathbf{f}_{e\ (12\times 1)} = \int_{-1}^{1} \mathbf{N}_e^T \mathbf{t} |\mathbf{J}| d\xi = \sum_{I=1}^{2} \alpha_I h(\xi_I)$$

where: $$h(\xi_I) = \mathbf{N}_e^T(\xi_I, \eta_I)\mathbf{t}|\mathbf{J}(\xi_I, \eta_I)|$$

Element Quantities and Identities

- Element CP weight vector

$$\mathbf{w}_e = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{bmatrix}^T$$
$$= \begin{bmatrix} 1 & 1 & 0.7071 & 0.7071 & 1 & 1 \end{bmatrix}^T$$

- Define: $\mathbf{R} = \begin{bmatrix} R_1 & R_2 \end{bmatrix}$
- A point on Γ_e is:

$$\mathbf{S}(u) = \begin{bmatrix} x & y \end{bmatrix} = \mathbf{R}\mathbf{C}_t$$

where:

$$\mathbf{C}_t = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}$$

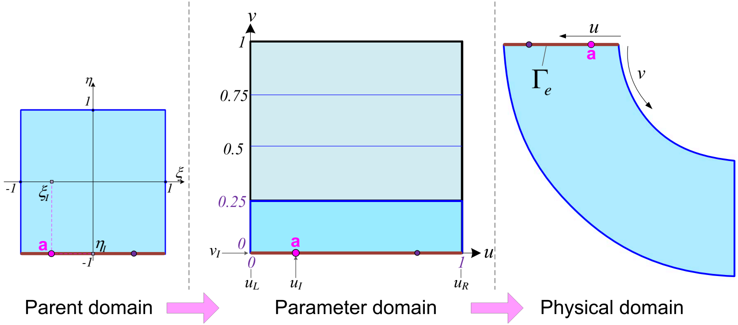
- Differentiate w.r.t ξ to get Jacobian

$$\mathbf{J} = \begin{bmatrix} x_{,\xi} & y_{,\xi} \end{bmatrix} = \mathbf{R}_{,\xi}\mathbf{C}_t$$

- N-matrix

$$\mathbf{N}_{e\ (2\times 12)} = \begin{bmatrix} R_1 & 0 & R_2 & 0 & \ldots & R_6 & 0 \\ 0 & R_1 & 0 & R_2 & \ldots & 0 & R_6 \end{bmatrix}$$

Gauss Point Mapping

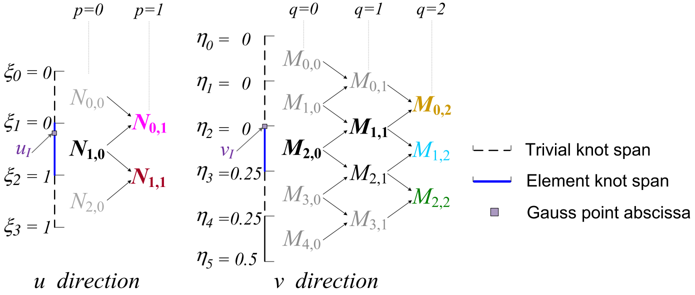


- Illustrative Gauss point **a**: $\xi_I = -0.5774, \alpha_I = 1$
- u direction knot span: $u_L = 0, u_R = 1$
- Mapping from $[-1, 1]$ to Γ_e

$$u = \frac{1}{2}(1-\xi)u_L + \frac{1}{2}(1+\xi)u_R$$

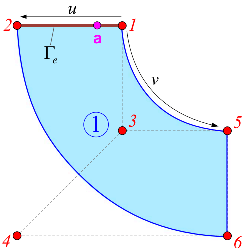
  - $u_I = 0.2113$ (after mapping)
  - $v_I = 0$ (since $v = 0$ on Γ_e)

Basis Function and Derivative in Each Direction

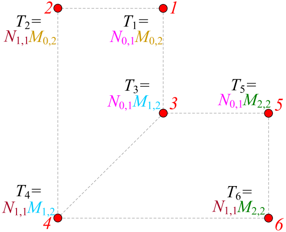


- Local knot vector for element ①
  - u direction: $\{\xi_0, \xi_1, \xi_2, \xi_3\}$
  - v direction: $\{\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$
- Basis function
  - u direction: $N_{0,1}(u_I) = 0.7887, N_{1,1}(u_I) = 0.2113$
  - v direction: $M_{0,2}(v_I) = 1, M_{1,2}(v_I) = 0, M_{2,2}(v_I) = 0$
- Derivatives w.r.t u
  - u direction: $(N_{0,1})_{,u}(u_I) = -0.5, (N_{1,1})_{,u}(u_I) = 0.5$
  - v direction: $(M_{0,2})_{,u}(v_I) = (M_{1,2})_{,u}(v_I) = (M_{2,2})_{,u}(v_I) = 0$

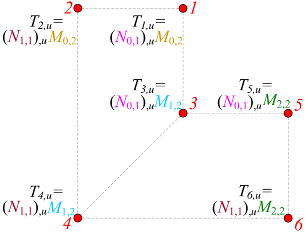
Tensor Product of Basis Function and Derivative



Element Control Net



Shape function tensor product



Shape function derivative tensor product

- Basis tensor product vector

$$\mathbf{T}_e = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 \end{bmatrix}$$
$$= \begin{bmatrix} 0.7887 & 0.2113 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Basis derivative tensor product vector

$$(\mathbf{T}_e)_{,u} = \begin{bmatrix} (T_1)_{,u} & (T_2)_{,u} & (T_3)_{,u} & (T_4)_{,u} & (T_5)_{,u} & (T_6)_{,u} \end{bmatrix}$$
$$= \begin{bmatrix} -0.5 & 0.5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element Force Vector

- Plug into shape function and derivative forluma

$$R_1 = 0.7887,\ R_2 = 0.2113,\ R_3 = R_4 = R_5 = R_6 = 0$$

$$R_{1,\xi} = -0.5,\ R_{2,\xi} = 0.5,\ R_{3,\xi} = R_{4,\xi} = R_{5,\xi} = R_{6,\xi} = 0$$

- Jacobian and its determinant

$$\mathbf{J} = \begin{bmatrix} R_{1,\xi} & R_{2,\xi} \end{bmatrix} \mathbf{C}_t = \begin{bmatrix} -0.5 & 0 \end{bmatrix},\ |\mathbf{J}| = 0.5$$

- N-matrix

$$\mathbf{N}_{e\ (2\times12)} = \begin{bmatrix} 0.7887 & 0 & 0.2113 & 0 & 0 & ... & 0 \\ 0 & 0.7887 & 0 & 0.2113 & 0 & ... & 0 \end{bmatrix}$$

- Perform previous steps for all 2 Gauss points and do summation to get element force vector in element ①

$$\mathbf{f}_{e\ (12\times1)} = \sum_{I=1}^{2} \alpha_I \mathbf{N}_e^T \mathbf{t} |\mathbf{J}| = \begin{bmatrix} 0 & -0.5 & 0 & -0.5 & 0 & ... & 0 \end{bmatrix}^T$$

- Assemble to get the global force vector

$$\mathbf{f}_{\ (36\times1)} = \sum_{e=1}^{4} \mathbf{L}_{e\ (36\times12)}^T \mathbf{f}_{e(12\times1)} = \begin{bmatrix} 0 & -0.5 & 0 & -0.5 & 0 & ... & 0 \end{bmatrix}^T$$

Solve System of Equations

- Reduced system of equations reads

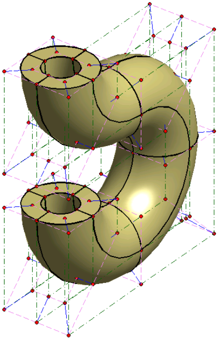
$$\widetilde{\mathbf{K}}\,\widetilde{\mathbf{d}} = \widetilde{\mathbf{f}}$$

  - $\widetilde{\mathbf{K}}_{(32\times32)}$ is global stiffness matrix $\mathbf{K}$ with $33, 34, 35, 36$-th rows and columns crossed out
  - $\widetilde{\mathbf{d}}_{(32\times1)}$ is global displacement vector $\mathbf{d}$ with $33, 34, 35, 36$-th rows (displacement=0) crossed out
  - $\widetilde{\mathbf{f}}_{(32\times1)}$ is global force vector $\mathbf{f}$ with $33, 34, 35, 36$-th rows crossed out

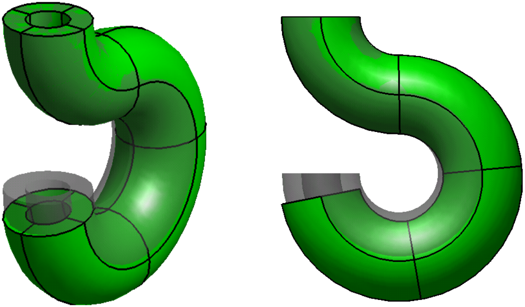
- Solve to get

$$\begin{aligned}\mathbf{d}_{(36\times1)} = [\,&0.2130 \quad -8.4350 \quad 0.1371 \quad -13.2891 \quad 4.8924 \quad -8.3848 \\ &9.9514 \quad -12.9467 \quad 5.1264 \quad -3.6956 \quad 9.4046 \quad -3.7714 \\ &5.3102 \quad 0.0892 \quad 8.1599 \quad 4.3904 \quad 2.1933 \quad 0.5556 \\ &2.2689 \quad 2.9412 \quad 0.5388 \quad 1.0210 \quad -1.3833 \quad 1.4935 \\ &0.3545 \quad 0.0356 \quad -0.1391 \quad 0.1097 \quad -0.0818 \quad -0.1973 \\ &0.0847 \quad -0.1301 \quad 0 \quad 0 \quad 0 \quad 0\,]^T \times 10^{-10}\end{aligned}$$

Hook: 3D example



Hook under end load



Before and after deformation